Day 1 Problem) a) In free space, Maxwell's first equation is $\boldsymbol{\nabla}\cdot(\varepsilon_0\boldsymbol{E})=0$. Application to the E-field of the plane-wave yields $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0$. Consequently, $\boldsymbol{k}\cdot\boldsymbol{E}_0=0$. This is the general relation between the k-vector and the magnitude $\boldsymbol{E}_0$ of the plane-wave's E-field.

b) In free space, Maxwell's fourth equation is $\boldsymbol{\nabla}\cdot(\mu_0\boldsymbol{H})=0$. Application to the H-field of the plane-wave yields $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0$. Consequently, $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$. This is the general relation between the k-vector and the magnitude $\boldsymbol{H}_0$ of the plane-wave's E-field.

c) Maxwell's second equation in free space is $\boldsymbol{\nabla}\times\boldsymbol{H}=\varepsilon_0\,\partial\boldsymbol{E}/\partial t$. Substitution for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the plane-wave expressions yields

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=-\mathrm{i}\omega\varepsilon_0\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\quad\rightarrow\quad\boldsymbol{k}\times\boldsymbol{H}_0=-\varepsilon_0\omega\boldsymbol{E}_0.$$

d) Maxwell's third equation in free space is $\boldsymbol{\nabla}\times\boldsymbol{E}=-\mu_0\,\partial\boldsymbol{H}/\partial t$. Substitution for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the plane-wave expressions yields

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=\mathrm{i}\omega\mu_0\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\quad\rightarrow\quad\boldsymbol{k}\times\boldsymbol{E}_0=\mu_0\omega\boldsymbol{H}_0.$$

e) From part (c) we know that $\boldsymbol{E}_0=-\boldsymbol{k}\times\boldsymbol{H}_0/(\varepsilon_0\omega)$. Substitution in the result obtained in part (d) then yields

$$-\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H}_0)/(\varepsilon_0\omega)=\mu_0\omega\boldsymbol{H}_0\qquad\rightarrow\qquad(\boldsymbol{k}\cdot\boldsymbol{H}_0)\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}_0=-\mu_0\varepsilon_0\omega^2\boldsymbol{H}_0.$$

Now, in part (b) we found that $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$. Therefore, the first term on the left-hand side of the preceding equation disappears, and we are left with $(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}_0=\mu_0\varepsilon_0\omega^2\boldsymbol{H}_0$. Dropping $\boldsymbol{H}_0$ from both sides of this equation yields

$$\boldsymbol{k}\cdot\boldsymbol{k}=(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')=({k'}^2-{k''}^2)+2\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{k}''=\mu_0\varepsilon_0\omega^2=(\omega/c)^2.$$

This is the general relation between the wave-vector $\boldsymbol{k}$ and the frequency ω of a plane-wave in free space.

Day 2 Problem)

a) $\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}} \;\rightarrow\; \varepsilon_o\boldsymbol{\nabla}\cdot\boldsymbol{E}=0 \;\rightarrow\; \partial E_z/\partial z=\partial[E_o\cos(k_o y-\omega_o t)]/\partial z=0.$ (1)

$\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}+\partial\boldsymbol{D}/\partial t \;\rightarrow\; -\partial H_x/\partial y=\varepsilon_o\partial E_z/\partial t$

$\rightarrow\; H_o k_o\sin(k_o y-\omega_o t)=\varepsilon_o E_o\omega_o\sin(k_o y-\omega_o t) \;\rightarrow\; H_o k_o=\varepsilon_o E_o\omega_o.$ (2)

$\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t \;\rightarrow\; \partial E_z/\partial y=-\mu_o\partial H_x/\partial t$

$\rightarrow\; -E_o k_o\sin(k_o y-\omega_o t)=-\mu_o H_o\omega_o\sin(k_o y-\omega_o t) \;\rightarrow\; E_o k_o=\mu_o H_o\omega_o.$ (3)

$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \;\rightarrow\; \mu_o\boldsymbol{\nabla}\cdot\boldsymbol{H}=0 \;\rightarrow\; \partial H_x/\partial x=\partial[H_o\cos(k_o y-\omega_o t)]/\partial x=0.$ (4)

It is seen that Maxwell's 1st and 4th equations are already satisfied. As for the 2nd and 3rd equations, we note that Eq.(2) above yields $E_o/H_o=k_o/(\varepsilon_o\omega_o)$, whereas Eq.(3) yields $E_o/H_o=\mu_o\omega_o/k_o$. Consequently, we must have $k_o/(\varepsilon_o\omega_o)=\mu_o\omega_o/k_o$, which yields $k_o=\omega_o/c$. Substitution into either Eq.(2) or Eq.(3) now reveals that $E_o/H_o=Z_o$.

b) The discontinuity of $D_\perp=\varepsilon_o E_z$ at each surface is equal to the surface charge-density at that surface, that is,

$$\sigma_s(x,y,z=\pm\tfrac{1}{2}d,t)=\mp\,\varepsilon_o E_o\cos(k_o y-\omega_o t). \tag{5}$$

Similarly, the discontinuity of $H_{||}=H_x$ at each surface is equal to the surface current-density at the corresponding surface, with the current's direction being perpendicular to that of the H-field. We thus have

$$\boldsymbol{J}_s(x,y,z=\pm\tfrac{1}{2}d,t)=\mp\,H_o\cos(k_o y-\omega_o t)\hat{\boldsymbol{y}}. \tag{6}$$

c) At each surface, the charge-current continuity equation $\boldsymbol{\nabla}\cdot\boldsymbol{J}+\partial\rho/\partial t=0$ reduces to $\partial J_{sy}/\partial y+\partial\sigma_s/\partial t=0$. With the help of Eqs.(5) and (6), we write the continuity equation as follows:

$$\partial J_{sy}/\partial y+\partial\sigma_s/\partial t=\pm H_o k_o\sin(k_o y-\omega_o t)\mp\varepsilon_o E_o\omega_o\sin(k_o y-\omega_o t)$$
$$=\pm(H_o k_o-\varepsilon_o E_o\omega_o)\sin(k_o y-\omega_o t)=0. \tag{7}$$

In the last line of the above equation, we have used Eq.(2) to set $H_o k_o$ equal to $\varepsilon_o E_o\omega_o$.