Solution to Problem 1) The k-vector in free space is aligned with $\hat{\boldsymbol{z}}$ and has magnitude $k_0 = \omega_0/c$. The $\boldsymbol{E}$ and $\boldsymbol{H}$ field amplitudes are $|E_0|\exp(\mathrm{i}\varphi_0)\,\hat{\boldsymbol{x}}$ and $Z_0^{-1}|E_0|\exp(\mathrm{i}\varphi_0)\,\hat{\boldsymbol{y}}$. We thus have

a) $$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\{E_0\hat{\boldsymbol{x}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_0 t)]\} = |E_0|\cos(k_0 z-\omega_0 t+\varphi_0)\,\hat{\boldsymbol{x}}, \tag{1a}$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \mathrm{Re}\{H_0\hat{\boldsymbol{y}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_0 t)]\} = Z_0^{-1}|E_0|\cos(k_0 z-\omega_0 t+\varphi_0)\,\hat{\boldsymbol{y}}. \tag{1b}$$

b) The rate of flow of electromagnetic (EM) energy per unit area per unit time is given by the Poynting vector, as follows:

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t) = Z_0^{-1}|E_0|^2\cos^2(k_0 z-\omega_0 t+\varphi_0)\,\hat{\boldsymbol{z}}$$
$$= \frac{|E_0|^2}{2Z_0}\{1+\cos[2(k_0 z-\omega_0 t+\varphi_0)]\}\hat{\boldsymbol{z}}. \tag{2}$$

At $t=t_0$, the values of the Poynting vector at P_1 and P_2 are $S(0,0,z_1,t_0)$ and $S(0,0,z_2,t_0)$, respectively. Therefore, the difference between the rates of energy inflow and outflow is

$$S_1-S_2 = ½(|E_0|^2/Z_0)\{\cos[2(k_0 z_1-\omega_0 t_0+\varphi_0)]-\cos[2(k_0 z_2-\omega_0 t_0+\varphi_0)]\}$$
$$= -(|E_0|^2/Z_0)\sin[k_0(z_1-z_2)]\sin[k_0(z_1+z_2)-2\omega_0 t_0+2\varphi_0]. \tag{3}$$

The above difference between S_1 and S_2 is seen to vanish if the distance between z_1 and z_2 happens to be $z_2-z_1 = m\pi/k_0 = m\pi c/\omega_0 = m\lambda_0/2$, with m being an arbitrary integer.

c) The energy-density of the EM field in free space is $\mathcal{E}(\boldsymbol{r},t) = ½\varepsilon_0 E^2 + ½\mu_0 H^2$. Therefore, for the plane-wave described in part (a), we have

$$\mathcal{E}(\boldsymbol{r},t) = ½(\varepsilon_0+\mu_0/Z_0^2)|E_0|^2\cos^2(k_0 z-\omega_0 t+\varphi_0) = \varepsilon_0|E_0|^2\cos^2(k_0 z-\omega_0 t+\varphi_0)$$
$$= ½\varepsilon_0|E_0|^2\{1+\cos[2(k_0 z-\omega_0 t+\varphi_0)]\}. \tag{4}$$

The integrated energy-density between z_1 and z_2 is thus equal to the energy (per unit cross-sectional area) contained in the region between P_1 and P_2, namely,

$$\int_{z_1}^{z_2}\mathcal{E}(\boldsymbol{r},t)dz = ½\varepsilon_0|E_0|^2\int_{z_1}^{z_2}\{1+\cos[2(k_0 z-\omega_0 t+\varphi_0)]\}dz$$
$$= ½\varepsilon_0|E_0|^2(z_2-z_1)$$
$$+¼\varepsilon_0|E_0|^2 k_0^{-1}\{\sin[2(k_0 z_2-\omega_0 t+\varphi_0)]-\sin[2(k_0 z_1-\omega_0 t+\varphi_0)]\}$$
$$= ½\varepsilon_0|E_0|^2(z_2-z_1)$$
$$-½(\varepsilon_0 c/\omega_0)|E_0|^2\sin[k_0(z_1-z_2)]\cos[k_0(z_1+z_2)-2\omega_0 t+2\varphi_0]. \tag{5}$$

Differentiating the above expression with respect to time now yields the time-rate-of-change of the stored EM energy in the region between P_1 and P_2 (per unit cross-sectional area), as follows:

$$\frac{d}{dt}\int_{z_1}^{z_2}\mathcal{E}(\boldsymbol{r},t)dz = -\varepsilon_0 c|E_0|^2\sin[k_0(z_1-z_2)]\sin[k_0(z_1+z_2)-2\omega_0 t+2\varphi_0]. \tag{6}$$

Given that $\varepsilon_0 c = 1/Z_0$, a comparison of Eq.(3) with Eq.(6) reveals that the difference between S_1 and S_2 is fully accounted for in terms of the time-rate-of-change of the stored EM energy in the region between P_1 and P_2.

Solution to Problem 2) a) In the transparent dielectric medium, the k-vector is $\pm(n_0\omega/c)\hat{\boldsymbol{z}}$, and the $\boldsymbol{H}$ field magnitude is $n_0 E_0/Z_0$. Consequently, the incident and reflected fields are given by

$$\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t) = E_0 \exp[-\mathrm{i}(\omega/c)(n_0 z + ct)]\hat{\boldsymbol{x}}, \tag{1a}$$

$$\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t) = -(n_0 E_0/Z_0)\exp[-\mathrm{i}(\omega/c)(n_0 z + ct)]\hat{\boldsymbol{y}}. \tag{1b}$$

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},t) = \rho E_0 \exp[\mathrm{i}(\omega/c)(n_0 z - ct)]\hat{\boldsymbol{x}}, \tag{2a}$$

$$\boldsymbol{H}^{(\mathrm{r})}(\boldsymbol{r},t) = (n_0 \rho E_0/Z_0)\exp[\mathrm{i}(\omega/c)(n_0 z - ct)]\hat{\boldsymbol{y}}. \tag{2b}$$

In the absorptive substrate, the k-vector is complex-valued, that is, $\boldsymbol{k} = -(n + \mathrm{i}\kappa)(\omega/c)\hat{\boldsymbol{z}}$, and the $\boldsymbol{H}$ field amplitude is $(n + \mathrm{i}\kappa)/Z_0$ times that of the $\boldsymbol{E}$ field. Therefore,

$$\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t) = \tau E_0 \exp\{-\mathrm{i}(\omega/c)[(n + \mathrm{i}\kappa)z + ct]\}\hat{\boldsymbol{x}}, \tag{3a}$$

$$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = -(n + \mathrm{i}\kappa)(\tau E_0/Z_0)\exp\{-\mathrm{i}(\omega/c)[(n + \mathrm{i}\kappa)z + ct]\}\hat{\boldsymbol{y}}. \tag{3b}$$

b) Matching the boundary conditions means enforcing the continuity of the $\boldsymbol{E}_\parallel$ and $\boldsymbol{H}_\parallel$ at $z = 0$. We will have

$$E_x^{(\mathrm{i})}(z = 0^+) + E_x^{(\mathrm{r})}(z = 0^+) = E_x^{(\mathrm{t})}(z = 0^-) \quad \rightarrow \quad E_0 + \rho E_0 = \tau E_0, \tag{4a}$$

$$H_y^{(\mathrm{i})}(z = 0^+) + H_y^{(\mathrm{r})}(z = 0^+) = H_y^{(\mathrm{t})}(z = 0^-) \quad \rightarrow \quad n_0 E_0 - n_0 \rho E_0 = \tau(n + \mathrm{i}\kappa)E_0. \tag{4b}$$

Solving the above equations for ρ and τ, we find

$$\rho = \frac{n_0 - (n+\mathrm{i}\kappa)}{n_0 + (n+\mathrm{i}\kappa)}, \tag{5a}$$

$$\tau = 1 + \rho = \frac{2n_0}{n_0 + (n+\mathrm{i}\kappa)}. \tag{5b}$$

c) The time-averaged rate of EM energy flow per unit area per unit time is $\langle \boldsymbol{S}(\boldsymbol{r},t)\rangle = ½\mathrm{Re}(\boldsymbol{E} \times \boldsymbol{H}^*)$. The corresponding entities for the incident, reflected, and transmitted beams are

$$\begin{aligned}\langle \boldsymbol{S}^{(\mathrm{i})}(\boldsymbol{r},t)\rangle &= -½\mathrm{Re}\{E_0 \exp[-\mathrm{i}(\omega/c)(n_0 z + ct)]\,(n_0 E_0^*/Z_0)\exp[\mathrm{i}(\omega/c)(n_0 z + ct)]\}\hat{\boldsymbol{z}} \\ &= -½ n_0 Z_0^{-1}|E_0|^2\hat{\boldsymbol{z}}.\end{aligned} \tag{6}$$

$$\langle \boldsymbol{S}^{(\mathrm{r})}(\boldsymbol{r},t)\rangle = ½ n_0 Z_0^{-1}|\rho E_0|^2\hat{\boldsymbol{z}}. \tag{7}$$

$$\begin{aligned}\langle \boldsymbol{S}^{(\mathrm{t})}(\boldsymbol{r},t)\rangle &= -½\mathrm{Re}\{\tau E_0 \exp(-\mathrm{i}(\omega/c)[(n + \mathrm{i}\kappa)z + ct]) \\ &\qquad \times (n - \mathrm{i}\kappa)\,(\tau^* E_0^*/Z_0)\exp(\mathrm{i}(\omega/c)[(n - \mathrm{i}\kappa)z + ct])\}\hat{\boldsymbol{z}} \\ &= -½ n Z_0^{-1}|\tau E_0|^2 \exp(2\kappa\omega z/c)\,\hat{\boldsymbol{z}}.\end{aligned} \tag{8}$$

d) The energy balance equation at $z = 0$ may thus be written as follows:

$$n_0 - n_0|\rho|^2 = n|\tau|^2 \qquad \rightarrow \qquad |\rho|^2 + \left(\frac{n}{n_0}\right)|\tau|^2 = 1. \tag{9}$$

To confirm the above energy balance equation, note that

$$|\rho|^2 = \rho\rho^* = \frac{(n_0-n-\mathrm{i}\kappa)(n_0-n+\mathrm{i}\kappa)}{(n_0+n+\mathrm{i}\kappa)(n_0+n-\mathrm{i}\kappa)} = \frac{(n_0-n)^2+\kappa^2}{(n_0+n)^2+\kappa^2}, \tag{10a}$$

$$|\tau|^2 = \tau\tau^* = \frac{4n_0^2}{(n_0+n)^2+\kappa^2}. \tag{10b}$$

Consequently,

$$|\rho|^2 + \left(\frac{n}{n_0}\right)|\tau|^2 = \frac{(n_0-n)^2+\kappa^2}{(n_0+n)^2+\kappa^2} + \frac{4nn_0}{(n_0+n)^2+\kappa^2} = \frac{(n_0+n)^2+\kappa^2}{(n_0+n)^2+\kappa^2} = 1. \tag{11}$$

The energy absorbed in the substrate is thus seen to be precisely equal to the difference between the incident and reflected energies.
