Spring 2017 Written Comprehensive Exam
Opti 501

Solution to Problem 1:

a) Denoting the wave-number by $k_0 = \omega/c$, and the normalized k-vector by $\boldsymbol{\sigma} = \boldsymbol{k}/k_0$, we write

$$\boldsymbol{E}(\boldsymbol{r},t) = E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}.$$

$$Z_0\boldsymbol{H}_0 = \boldsymbol{\sigma}\times\boldsymbol{E}_0 \quad\rightarrow\quad Z_0\boldsymbol{H}_0 = n(\omega)E_0(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}}) \quad\rightarrow\quad \boldsymbol{H}_0 = n(\omega)E_0\hat{\boldsymbol{y}}/Z_0$$

$$\rightarrow\quad \boldsymbol{H}(\boldsymbol{r},t) = \left[\frac{n(\omega)E_0}{Z_0}\right]\hat{\boldsymbol{y}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}.$$

b) $n(\omega) = \sqrt{\varepsilon(\omega)} \quad\rightarrow\quad \varepsilon(\omega) = n^2(\omega).$

$\varepsilon(\omega) = 1 + \chi(\omega) \quad\rightarrow\quad \chi(\omega) = n^2(\omega) - 1.$

c) $\boldsymbol{P}(\boldsymbol{r},t) = \varepsilon_0\chi(\omega)E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}$

$$= \varepsilon_0[n^2(\omega) - 1]E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}.$$

$$\rho_{\text{bound}}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t) = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = 0.$$

$$\boldsymbol{J}_{\text{bound}}(\boldsymbol{r},t) = \frac{\partial \boldsymbol{P}(\boldsymbol{r},t)}{\partial t} = -\mathrm{i}\omega\varepsilon_0[n^2(\omega) - 1]E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}.$$

The actual $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{P}, \boldsymbol{J}_{\text{bound}}$ are, of course, given by the *real parts* of the above expressions.

Solution to Problem 2:

a) In the free-space region, the incident k-vector is $\boldsymbol{k}^{(i)} = (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}})$. The E and H fields may then be written in terms of $\boldsymbol{k}^{(i)}$, ω, and the E-field amplitude E_0, as follows:

$$\boldsymbol{E}^{(i)}(\boldsymbol{r},t) = \mathrm{Re}\{E_0(\cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}})\exp[i(\boldsymbol{k}^{(i)}\cdot\boldsymbol{r} - \omega t)]\},$$

$$\boldsymbol{H}^{(i)}(\boldsymbol{r},t) = \mathrm{Re}\{Z_0^{-1}E_0\hat{\boldsymbol{y}}\exp[i(\boldsymbol{k}^{(i)}\cdot\boldsymbol{r} - \omega t)]\}.$$

b) For the reflected beam, the k-vector is $\boldsymbol{k}^{(r)} = (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}})$, and the E and H fields, expressed as functions of $\boldsymbol{k}^{(r)}$, ω, the Fresnel reflection coefficient ρ_p, and the incident E-field amplitude E_0, are

$$\boldsymbol{E}^{(r)}(\boldsymbol{r},t) = \mathrm{Re}\{\rho_p E_0(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}})\exp[i(\boldsymbol{k}^{(r)}\cdot\boldsymbol{r} - \omega t)]\},$$

$$\boldsymbol{H}^{(r)}(\boldsymbol{r},t) = -\,\mathrm{Re}\{Z_0^{-1}\rho_p E_0\hat{\boldsymbol{y}}\exp[i(\boldsymbol{k}^{(r)}\cdot\boldsymbol{r} - \omega t)]\}.$$

c) For the transmitted beam, the k-vector is $\boldsymbol{k}^{(t)} = (\omega/c)[\sin\theta\,\hat{\boldsymbol{x}} + \sqrt{\varepsilon(\omega) - \sin^2\theta}\,\hat{\boldsymbol{z}}]$. This is derived from the continuity of k_x across the interface, and from the dispersion relation of the plasma, namely, $k_x^2 + k_z^2 = (\omega/c)^2\mu(\omega)\varepsilon(\omega)$. The E and H fields, written in terms of $\boldsymbol{k}^{(t)}$, ω, the Fresnel transmission coefficient τ_p, and the incident E-field amplitude E_0, are

$$\boldsymbol{E}^{(t)}(\boldsymbol{r},t) = \mathrm{Re}\left\{\tau_p E_0\cos\theta\left(\hat{\boldsymbol{x}} - \frac{\sin\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}}\hat{\boldsymbol{z}}\right)\exp[i(\boldsymbol{k}^{(t)}\cdot\boldsymbol{r} - \omega t)]\right\},$$

$$\boldsymbol{H}^{(t)}(\boldsymbol{r},t) = \mathrm{Re}\left\{\frac{\tau_p\varepsilon(\omega)E_0\cos\theta}{Z_0\sqrt{\varepsilon(\omega)-\sin^2\theta}}\hat{\boldsymbol{y}}\exp[i(\boldsymbol{k}^{(t)}\cdot\boldsymbol{r} - \omega t)]\right\}.$$

In deriving the above expressions, we used the constraints imposed by Maxwell's 1st and 3rd equations, namely, $\boldsymbol{k}^{(t)}\cdot\boldsymbol{E}^{(t)} = k_x^{(t)}E_x^{(t)} + k_z^{(t)}E_z^{(t)} = 0$ and $\boldsymbol{k}^{(t)}\times\boldsymbol{E}^{(t)} = \mu_0\mu(\omega)\omega\boldsymbol{H}^{(t)}$.

d) The tangential components $E_x^{(i)}, E_x^{(r)}, E_x^{(t)}$ of the E-field must satisfy the continuity condition at the interface, as do the tangential components $H_y^{(i)}, H_y^{(r)}, H_y^{(t)}$ of the H-field. Therefore,

$\boldsymbol{E}_\parallel$ continuity: $E_0\cos\theta + \rho_p E_0\cos\theta = \tau_p E_0\cos\theta \quad\rightarrow\quad 1 + \rho_p = \tau_p.$

$\boldsymbol{H}_\parallel$ continuity: $Z_0^{-1}E_0 - Z_0^{-1}\rho_p E_0 = \dfrac{\tau_p\varepsilon(\omega)E_0\cos\theta}{Z_0\sqrt{\varepsilon(\omega)-\sin^2\theta}} \quad\rightarrow\quad 1 - \rho_p = \dfrac{\tau_p\varepsilon(\omega)\cos\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}}.$

Solving the above equations, we find $\rho_p = \dfrac{\sqrt{\varepsilon(\omega)-\sin^2\theta}-\varepsilon(\omega)\cos\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}+\varepsilon(\omega)\cos\theta}$ and $\tau_p = \dfrac{2\sqrt{\varepsilon(\omega)-\sin^2\theta}}{\sqrt{\varepsilon(\omega)-\sin^2\theta}+\varepsilon(\omega)\cos\theta}$.

e) Since $\varepsilon(\omega)$ is real-valued and negative, ρ_p may be written as follows:

$$\rho_p = \frac{i\sqrt{|\varepsilon(\omega)|+\sin^2\theta}+|\varepsilon(\omega)|\cos\theta}{i\sqrt{|\varepsilon(\omega)|+\sin^2\theta}-|\varepsilon(\omega)|\cos\theta}$$

Thus ρ_p is seen to be the ratio of a complex number to its conjugate, which has a magnitude of 1. Since $|\rho_p| = 1$, the reflectivity is 100%. This does not contradict the existence of electromagnetic waves within the plasma, because the time-averaged Poynting vector of the plane-wave inside the plasma, like that of an evanescent wave, has a vanishing z-component.