Solution to Problem 1) a) Setting $\rho_{\text{free}} = 0$, $\boldsymbol{J}_{\text{free}} = 0$, $\boldsymbol{D} = \varepsilon_0\varepsilon\boldsymbol{E}$, and $\boldsymbol{B} = \mu_0\mu\boldsymbol{H}$, Maxwell's equations can be simplified as follows:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\varepsilon\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = 0 \quad\rightarrow\quad \boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r}) = 0.$$

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t) = \partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t = -\mathrm{i}\omega\varepsilon_0\varepsilon\boldsymbol{E}(\boldsymbol{r},t) \quad\rightarrow\quad \boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) = -\mathrm{i}\omega\varepsilon_0\varepsilon\boldsymbol{E}(\boldsymbol{r}).$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t) = -\,\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t = \mathrm{i}\omega\mu_0\mu\boldsymbol{H}(\boldsymbol{r},t) \quad\rightarrow\quad \boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r}) = \mathrm{i}\omega\mu_0\mu\boldsymbol{H}(\boldsymbol{r}).$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\mu\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r},t) = 0 \quad\rightarrow\quad \boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r}) = 0.$$

b) Taking the curl of the 3rd equation, then substituting for $\boldsymbol{\nabla}\times\boldsymbol{H}$ from the 2nd equation, yields

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r}) = \mathrm{i}\omega\mu_0\mu\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) = \mu_0\varepsilon_0\mu\varepsilon\omega^2\boldsymbol{E}(\boldsymbol{r}) = (\sqrt{\mu\varepsilon}\,\omega/c)^2\boldsymbol{E}(\boldsymbol{r}).$$

From the 1st equation, we know that $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$. Therefore,

$$\boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r})] - \boldsymbol{\nabla}^2\boldsymbol{E}(\boldsymbol{r}) = (\sqrt{\mu\varepsilon}\,\omega/c)^2\boldsymbol{E}(\boldsymbol{r}) \quad\rightarrow\quad \boldsymbol{\nabla}^2\boldsymbol{E}(\boldsymbol{r}) + (\sqrt{\mu\varepsilon}\,\omega/c)^2\boldsymbol{E}(\boldsymbol{r}) = 0.$$

c) Taking the curl of the 2nd equation, then substituting for $\boldsymbol{\nabla}\times\boldsymbol{E}$ from the 3rd equation, yields

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) = -\mathrm{i}\omega\varepsilon_0\varepsilon\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r}) = \mu_0\varepsilon_0\mu\varepsilon\omega^2\boldsymbol{H}(\boldsymbol{r}) = (\sqrt{\mu\varepsilon}\,\omega/c)^2\boldsymbol{H}(\boldsymbol{r}).$$

From the 4th equation, we know that $\boldsymbol{\nabla}\cdot\boldsymbol{H} = 0$. Therefore,

$$\boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r})] - \boldsymbol{\nabla}^2\boldsymbol{H}(\boldsymbol{r}) = (\sqrt{\mu\varepsilon}\,\omega/c)^2\boldsymbol{H}(\boldsymbol{r}) \quad\rightarrow\quad \boldsymbol{\nabla}^2\boldsymbol{H}(\boldsymbol{r}) + (\sqrt{\mu\varepsilon}\,\omega/c)^2\boldsymbol{H}(\boldsymbol{r}) = 0.$$

Solution to Problem 2) a) The various plane-waves are all linearly-polarized along the x-axis, with their magnetic fields aligned with either $\hat{\boldsymbol{y}}$ or $-\hat{\boldsymbol{y}}$. The k-vector is either along $\hat{\boldsymbol{z}}$ or $-\hat{\boldsymbol{z}}$, its magnitude being $k_0 = \omega/c$ in air, n_0k_0 in the dielectric layer, and $(n + \mathrm{i}\kappa)k_0$ in the metallic substrate. The $\boldsymbol{E}$ and $\boldsymbol{H}$ fields in the three regions are

Incident beam:

$$\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t) = E_0\hat{\boldsymbol{x}}\exp[-\mathrm{i}(\omega/c)(z + ct)],$$

$$\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t) = -(E_0/Z_0)\hat{\boldsymbol{y}}\exp[-\mathrm{i}(\omega/c)(z + ct)].$$

Reflected beam:

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},t) = \rho E_0\hat{\boldsymbol{x}}\exp[\mathrm{i}(\omega/c)(z - ct)],$$

$$\boldsymbol{H}^{(\mathrm{r})}(\boldsymbol{r},t) = \rho(E_0/Z_0)\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(z - ct)].$$

Inside dielectric:

$$\boldsymbol{E}^{(\mathrm{A})}(\boldsymbol{r},t) = E_A\hat{\boldsymbol{x}}\exp[-\mathrm{i}(\omega/c)(n_0z + ct)],$$

$$\boldsymbol{H}^{(\mathrm{A})}(\boldsymbol{r},t) = -(n_0E_A/Z_0)\hat{\boldsymbol{y}}\exp[-\mathrm{i}(\omega/c)(n_0z + ct)].$$

$$\boldsymbol{E}^{(\mathrm{B})}(\boldsymbol{r},t) = E_B\hat{\boldsymbol{x}}\exp[\mathrm{i}(\omega/c)(n_0z - ct)],$$

$$\boldsymbol{H}^{(\mathrm{B})}(\boldsymbol{r},t) = (n_0E_B/Z_0)\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(n_0z - ct)].$$

Inside metal:

$$\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t) = \tau E_0\hat{\boldsymbol{x}}\exp\{-\mathrm{i}(\omega/c)[(n + \mathrm{i}\kappa)z + ct]\},$$

$$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = -[(n + \mathrm{i}\kappa)\tau E_0/Z_0]\hat{\boldsymbol{y}}\exp\{-\mathrm{i}(\omega/c)[(n + \mathrm{i}\kappa)z + ct]\}.$$

b) Matching the boundary conditions at the air-glass interface ($z = 0$) yields

Continuity of $E_\parallel$: $E_0 + \rho E_0 = E_A + E_B \quad \rightarrow \quad (1+\rho)E_0 = E_A + E_B.$

Continuity of $H_\parallel$: $-\frac{E_0}{Z_0} + \frac{\rho E_0}{Z_0} = -\frac{n_0 E_A}{Z_0} + \frac{n_0 E_B}{Z_0} \quad \rightarrow \quad (1-\rho)E_0 = n_0(E_A - E_B).$

Dividing the above equations eliminates E_0, yielding ρ in terms of $\rho_1 = (1-n_0)/(1+n_0)$ and E_B/E_A, as follows:

$$\frac{1-\rho}{1+\rho} = n_0 \frac{1-(E_B/E_A)}{1+(E_B/E_A)} \quad \rightarrow \quad \rho = \frac{[(1-n_0)/(1+n_0)] + (E_B/E_A)}{1 + [(1-n_0)/(1+n_0)](E_B/E_A)} = \frac{\rho_1 + (E_B/E_A)}{1 + \rho_1(E_B/E_A)}.$$

Matching the boundary conditions at the glass-metal interface ($z = -d_0$) yields

Continuity of $E_\parallel$: $E_A \exp(\mathrm{i}\omega n_0 d_0/c) + E_B \exp(-\mathrm{i}\omega n_0 d_0/c) = \tau E_0 \exp[\mathrm{i}(\omega/c)(n+\mathrm{i}\kappa)d_0].$

Continuity of $H_\parallel$: $-(n_0 E_A/Z_0)\exp(\mathrm{i}\omega n_0 d_0/c) + (n_0 E_B/Z_0)\exp(-\mathrm{i}\omega n_0 d_0/c)$

$$= -[(n+\mathrm{i}\kappa)\tau E_0/Z_0]\exp[\mathrm{i}(\omega/c)(n+\mathrm{i}\kappa)d_0].$$

Dividing the above equations eliminates the transmission coefficient τ, yielding an expression for E_B/E_A in terms of $\rho_2 = [n_0 - (n+\mathrm{i}\kappa)]/[n_0 + (n+\mathrm{i}\kappa)] = |\rho_2|\exp(\mathrm{i}\varphi_2)$ and the round-trip phase $\varphi_0 = 2\omega n_0 d_0/c = 4\pi n_0 d_0/\lambda_0$, namely,

$$\frac{E_A \exp(\mathrm{i}\omega n_0 d_0/c) - E_B \exp(-\mathrm{i}\omega n_0 d_0/c)}{E_A \exp(\mathrm{i}\omega n_0 d_0/c) + E_B \exp(-\mathrm{i}\omega n_0 d_0/c)} = \frac{n+\mathrm{i}\kappa}{n_0} \quad \rightarrow \quad \frac{\exp(\mathrm{i}\varphi_0) - (E_B/E_A)}{\exp(\mathrm{i}\varphi_0) + (E_B/E_A)} = (n+\mathrm{i}\kappa)/n_0$$

$$\rightarrow \quad \frac{E_B}{E_A} = \frac{n_0 - (n+\mathrm{i}\kappa)}{n_0 + (n+\mathrm{i}\kappa)}\exp(\mathrm{i}\varphi_0) = \rho_2 \exp(\mathrm{i}\varphi_0) = |\rho_2|\exp[\mathrm{i}(\varphi_2 + \varphi_0)].$$

c) Substitution for E_B/E_A from the above equation into an earlier expression for ρ now yields

$$\rho = \frac{\rho_1 + |\rho_2|\exp[\mathrm{i}(\varphi_2+\varphi_0)]}{1 + \rho_1|\rho_2|\exp[\mathrm{i}(\varphi_2+\varphi_0)]}.$$

The reflectance of the coated metallic surface is obtained by squaring the magnitude of the above Fresnel reflection coefficient, that is,

$$R = |\rho|^2 = \rho\rho^* = \frac{\rho_1^2 + |\rho_2|^2 + 2\rho_1|\rho_2|\cos(\varphi_0+\varphi_2)}{1 + \rho_1^2|\rho_2|^2 + 2\rho_1|\rho_2|\cos(\varphi_0+\varphi_2)}.$$

d) The reflectance R is a function of the round-trip phase $\varphi_0 = 4\pi n_0 d_0/\lambda_0$, which may be adjusted by varying the thickness d_0 of the dielectric layer. The maximum and minimum values of R are determined by setting its derivative with respect to φ_0 equal to zero, that is,

$$\frac{dR}{d\varphi_0} = -\frac{2\rho_1|\rho_2|(1-\rho_1^2)(1-|\rho_2|^2)\sin(\varphi_0+\varphi_2)}{[1 + \rho_1^2|\rho_2|^2 + 2\rho_1|\rho_2|\cos(\varphi_0+\varphi_2)]^2} = 0 \quad \rightarrow \quad \sin(\varphi_0+\varphi_2) = 0.$$

Considering that $\rho_1 < 0$, the reflectance reaches a minimum when $\cos(\varphi_0+\varphi_2) = 1$, that is,

$$R_{\min} = \left(\frac{\rho_1 + |\rho_2|}{1 + \rho_1|\rho_2|}\right)^2.$$

The maximum reflectance is reached when $\cos(\varphi_0+\varphi_2) = -1$, that is,

$$R_{\max} = \left(\frac{\rho_1 - |\rho_2|}{1 - \rho_1|\rho_2|}\right)^2.$$