**Spring 2015 Written Comprehensive Exam**
**Opti 501**

**Solution to Problem 1**:

a) $\boldsymbol{E}^{(\text{total})} = \boldsymbol{E}^{(\text{inc})} + \boldsymbol{E}^{(\text{ref})} = E_o \hat{\boldsymbol{x}} \{\cos[(\omega/c)z - \omega t] - \cos[(\omega/c)z + \omega t]\} = 2E_o \hat{\boldsymbol{x}} \sin(\omega z/c)\sin(\omega t).$

$\boldsymbol{H}^{(\text{total})} = \boldsymbol{H}^{(\text{inc})} + \boldsymbol{H}^{(\text{ref})} = (E_o/Z_o)\hat{\boldsymbol{y}} \{\cos[(\omega/c)z - \omega t] + \cos[(\omega/c)z + \omega t]\} = 2(E_o/Z_o)\hat{\boldsymbol{y}} \cos(\omega z/c)\cos(\omega t).$

b) The $E$-field vanishes where $\sin(\omega z/c) = 0$, that is, $z = 0, -\lambda/2, -\lambda, -3\lambda/2, \ldots$ . Here $\lambda = 2\pi c/\omega$.
The $H$-field vanishes where $\cos(\omega z/c) = 0$, that is, $z = -\lambda/4, -3\lambda/4, -5\lambda/4, \ldots$ .

c) Energy density of the $E$-field: $\frac{1}{2}\varepsilon_o |\boldsymbol{E}|^2 = 2\varepsilon_o E_o^2 \sin^2(\omega z/c)\sin^2(\omega t).$

Energy density of the $H$-field: $\frac{1}{2}\mu_o |\boldsymbol{H}|^2 = 2\varepsilon_o E_o^2 \cos^2(\omega z/c)\cos^2(\omega t).$

d) $\boldsymbol{S}(z,t) = \boldsymbol{E}^{(\text{total})} \times \boldsymbol{H}^{(\text{total})} = (E_o^2/Z_o)\hat{\boldsymbol{z}} \sin(2\omega z/c)\sin(2\omega t).$

The $z$-dependence of the Poynting vector, $\sin(2\omega z/c) = \sin(4\pi z/\lambda)$, reveals that $\boldsymbol{S}(z,t)$ is zero at all integer multiples of $\lambda/4$. Therefore, where either the $E$-field or the $H$-field of the standing wave has a node, no energy flows at all. The energy only flows along $z$ in between these adjacent nodes, which are separated by intervals of $\Delta z = \lambda/4$. The time-dependence of the Poynting vector, $\sin(2\omega t)$, shows that energy flow along $z$ changes direction at twice the optical frequency $\omega$. There are periodic instants when the energy is entirely in the $E$-field, followed by instants when the energy is entirely in the $H$-field. In between, the energy moves either slightly to the right or slightly to the left along $z$, in order to maintain the $E$- and $H$-field energy profiles found in part (c).

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**Solution to Problem 2**:

a) Denoting the wave-number by $k_0 = \omega/c$, and the normalized $k$-vector by $\boldsymbol{\sigma} = \boldsymbol{k}/k_0$, we write

$$\boldsymbol{E}(\boldsymbol{r},t) = E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}.$$

$$Z_0\boldsymbol{H}_0 = \boldsymbol{\sigma}\times\boldsymbol{E}_0 \quad\rightarrow\quad Z_0\boldsymbol{H}_0 = n(\omega)E_0(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}}) \quad\rightarrow\quad \boldsymbol{H}_0 = n(\omega)E_0\hat{\boldsymbol{y}}/Z_0$$

$$\rightarrow\quad \boldsymbol{H}(\boldsymbol{r},t) = \left[\frac{n(\omega)E_0}{Z_0}\right]\hat{\boldsymbol{y}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}.$$

b) $$n(\omega) = \sqrt{\varepsilon(\omega)} \quad\rightarrow\quad \varepsilon(\omega) = n^2(\omega).$$

$$\varepsilon(\omega) = 1 + \chi(\omega) \quad\rightarrow\quad \chi(\omega) = n^2(\omega) - 1.$$

c) $$\boldsymbol{P}(\boldsymbol{r},t) = \varepsilon_0\chi(\omega)E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}$$

$$= \varepsilon_0[n^2(\omega) - 1]E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}.$$

$$\rho_{\text{bound}}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t) = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = 0.$$

$$\boldsymbol{J}_{\text{bound}}(\boldsymbol{r},t) = \frac{\partial \boldsymbol{P}(\boldsymbol{r},t)}{\partial t} = -\mathrm{i}\omega\varepsilon_0[n^2(\omega) - 1]E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}k_0[n(\omega)z - ct]\}.$$

The actual $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{P}, \boldsymbol{J}_{\text{bound}}$ are, of course, given by the *real parts* of the above expressions.