Spring 2014 Written Comprehensive Exam
Opti 501

Solution to Problem 1:

a) $$\boldsymbol{k} = k_0\hat{\boldsymbol{z}} = (\omega/c)\hat{\boldsymbol{z}}.$$

b) The beam is linearly-polarized if either $E_{x0} = 0$ or $E_{y0} = 0$ or $\varphi_{x0} = \varphi_{y0}$ or $\varphi_{x0} = \varphi_{y0} \pm \pi$. The beam is circularly-polarized if $E_{x0} = E_{y0}$ *and* $\varphi_{x0} - \varphi_{y0} = \pm\pi/2$. Under all other circumstances, the beam will be elliptically-polarized.

c) Starting with the assumption that the amplitude and phase of the *H*-field components are (H_{x0}, ψ_{x0}) and (H_{y0}, ψ_{y0}), we write

$$\boldsymbol{H}(\boldsymbol{r}, t) = H_{x0}\cos(k_0 z - \omega t + \psi_{x0})\,\hat{\boldsymbol{x}} + H_{y0}\cos(k_0 z - \omega t + \psi_{y0})\,\hat{\boldsymbol{y}}.$$

Maxwell's 3rd equation then yields

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \rightarrow \quad -\frac{\partial E_y}{\partial z}\hat{\boldsymbol{x}} + \frac{\partial E_x}{\partial z}\hat{\boldsymbol{y}} = -\mu_0\left(\frac{\partial H_x}{\partial t}\hat{\boldsymbol{x}} + \frac{\partial H_y}{\partial t}\hat{\boldsymbol{y}}\right).$$

Consequently,

$$-\frac{\partial E_y}{\partial z} = -\mu_0\frac{\partial H_x}{\partial t} \rightarrow \quad k_0 E_{y0}\sin(k_0 z - \omega t + \varphi_{y0}) = -\mu_0 H_{x0}\omega\sin(k_0 z - \omega t + \psi_{x0})$$

$$\rightarrow \quad (\omega/c)E_{y0}\sin(k_0 z - \omega t + \varphi_{y0}) = -\mu_0 H_{x0}\omega\sin(k_0 z - \omega t + \psi_{x0})$$

$$\rightarrow \quad H_{x0} = -E_{y0}/(\mu_0 c) = -E_{y0}/Z_0 \quad \text{and} \quad \psi_{x0} = \varphi_{y0}.$$

Similarly,

$$\frac{\partial E_x}{\partial z} = -\mu_0\frac{\partial H_y}{\partial t} \quad \rightarrow \quad -k_0 E_{x0}\sin(k_0 z - \omega t + \varphi_{x0}) = -\mu_0 H_{y0}\omega\sin(k_0 z - \omega t + \psi_{y0})$$

$$\rightarrow \quad (\omega/c)E_{x0}\sin(k_0 z - \omega t + \varphi_{x0}) = \mu_0 H_{y0}\omega\sin(k_0 z - \omega t + \psi_{y0})$$

$$\rightarrow \quad H_{y0} = E_{x0}/(\mu_0 c) = E_{x0}/Z_0 \quad \text{and} \quad \psi_{y0} = \varphi_{x0}.$$

d) Direct multiplication of the *E*-field into the *H*-field obtained in part (c) now yields

$$\boldsymbol{S}(\boldsymbol{r}, t) = Z_0^{-1}\left[E_{x0}\cos(k_0 z - \omega t + \varphi_{x0})\,\hat{\boldsymbol{x}} + E_{y0}\cos(k_0 z - \omega t + \varphi_{y0})\,\hat{\boldsymbol{y}}\right]$$
$$\times\left[-E_{y0}\cos(k_0 z - \omega t + \varphi_{y0})\,\hat{\boldsymbol{x}} + E_{x0}\cos(k_0 z - \omega t + \varphi_{x0})\,\hat{\boldsymbol{y}}\right]$$
$$= Z_0^{-1}\left[E_{x0}^2\cos^2(k_0 z - \omega t + \varphi_{x0}) + E_{y0}^2\cos^2(k_0 z - \omega t + \varphi_{y0})\right]\hat{\boldsymbol{z}}.$$

The Poynting vector $\boldsymbol{S}(\boldsymbol{r}, t)$ is the rate of flow of electromagnetic energy per unit area per unit time, evaluated at the point $\boldsymbol{r}$ in space and at the instant t of time. It *must* satisfy the energy continuity equation at *all* points $\boldsymbol{r}$ in space at *all* instants t in time.

e) For circular-polarization, we have $E_{x0} = E_{y0}$ and $\varphi_{x0} = \varphi_{y0} \pm \pi/2$. Therefore,

$$\boldsymbol{S}(\boldsymbol{r},t) = Z_0^{-1}E_{x0}^2[\cos^2(k_0 z - \omega t + \varphi_{x0}) + \sin^2(k_0 z - \omega t + \varphi_{x0})]\hat{\boldsymbol{z}} = Z_0^{-1}E_{x0}^2\,\hat{\boldsymbol{z}}.$$

Clearly, the above expression is independent of z and t. The electromagnetic energy thus flows uniformly and at the constant rate of E_{x0}^2/Z_0 along the z-axis

f) For a linearly-polarized beam, we will have

$$\boldsymbol{S}(\boldsymbol{r},t) = Z_0^{-1}\left(E_{x0}^2 + E_{y0}^2\right)\cos^2(k_0 z - \omega t + \varphi_{x0})\,\hat{\boldsymbol{z}}.$$

The above $\boldsymbol{S}$ obviously varies with both z and t. This means that at any given time, say, $t = t_0$, the energy crossing a plane perpendicular to the z-axis at z_1 is different from the energy crossing another perpendicular plane at z_2. Conservation of energy is not violated, however, because, unlike the case of circular-polarization, the energy stored in the E and H fields in the region between z_1 and z_2 is not constant in this case. Recall that Poynting's theorem in free-space requires that $\boldsymbol{\nabla}\cdot\boldsymbol{S} + \partial(½\varepsilon_0\boldsymbol{E}\cdot\boldsymbol{E} + ½\mu_0\boldsymbol{H}\cdot\boldsymbol{H})/\partial t = 0$. Consequently, the difference between the energy entering at $z = z_1$ and the energy leaving at $z = z_2$ is given to (or taken away from) the energy stored in the E and H fields in the space between z_1 and z_2.

Solution to Problem 2:

a) In the free-space region, the incident k-vector is $\boldsymbol{k}^{(i)} = (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}})$. The E and H fields may then be written in terms of $\boldsymbol{k}^{(i)}, \omega$, and the E-field amplitude E_0, as follows:

$$\boldsymbol{E}^{(i)}(\boldsymbol{r},t) = \mathrm{Re}\{E_0(\cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}})\exp[i(\boldsymbol{k}^{(i)}\cdot\boldsymbol{r} - \omega t)]\},$$

$$\boldsymbol{H}^{(i)}(\boldsymbol{r},t) = \mathrm{Re}\{Z_0^{-1}E_0\hat{\boldsymbol{y}}\exp[i(\boldsymbol{k}^{(i)}\cdot\boldsymbol{r} - \omega t)]\}.$$

b) For the reflected beam, the k-vector is $\boldsymbol{k}^{(r)} = (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}})$, and the E and H fields, expressed as functions of $\boldsymbol{k}^{(r)}, \omega$, the Fresnel reflection coefficient ρ_p, and the incident E-field amplitude E_0, are

$$\boldsymbol{E}^{(r)}(\boldsymbol{r},t) = \mathrm{Re}\{\rho_p E_0(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}})\exp[i(\boldsymbol{k}^{(r)}\cdot\boldsymbol{r} - \omega t)]\},$$

$$\boldsymbol{H}^{(r)}(\boldsymbol{r},t) = -\,\mathrm{Re}\{Z_0^{-1}\rho_p E_0\hat{\boldsymbol{y}}\exp[i(\boldsymbol{k}^{(r)}\cdot\boldsymbol{r} - \omega t)]\}.$$

c) For the transmitted beam, the k-vector is $\boldsymbol{k}^{(t)} = (\omega/c)\left[\sin\theta\,\hat{\boldsymbol{x}} + \sqrt{\varepsilon(\omega) - \sin^2\theta}\,\hat{\boldsymbol{z}}\right]$. This is derived from the continuity of k_x across the interface, and from the dispersion relation of the plasma, namely, $k_x^2 + k_z^2 = (\omega/c)^2\mu(\omega)\varepsilon(\omega)$. The E and H fields, written in terms of $\boldsymbol{k}^{(t)}, \omega$, the Fresnel transmission coefficient τ_p, and the incident E-field amplitude E_0, are

$$\boldsymbol{E}^{(t)}(\boldsymbol{r},t) = \mathrm{Re}\left\{\tau_p E_0\cos\theta\left(\hat{\boldsymbol{x}} - \frac{\sin\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}}\hat{\boldsymbol{z}}\right)\exp[i(\boldsymbol{k}^{(t)}\cdot\boldsymbol{r} - \omega t)]\right\},$$

$$\boldsymbol{H}^{(t)}(\boldsymbol{r},t) = \mathrm{Re}\left\{\frac{\tau_p\varepsilon(\omega)E_0\cos\theta}{Z_0\sqrt{\varepsilon(\omega)-\sin^2\theta}}\hat{\boldsymbol{y}}\exp[i(\boldsymbol{k}^{(t)}\cdot\boldsymbol{r} - \omega t)]\right\}.$$

In deriving the above expressions, we used the constraints imposed by Maxwell's 1st and 3rd equations, namely, $\boldsymbol{k}^{(t)}\cdot\boldsymbol{E}^{(t)} = k_x^{(t)}E_x^{(t)} + k_z^{(t)}E_z^{(t)} = 0$ and $\boldsymbol{k}^{(t)}\times\boldsymbol{E}^{(t)} = \mu_0\mu(\omega)\omega\boldsymbol{H}^{(t)}$.

d) The tangential components $E_x^{(i)}, E_x^{(r)}, E_x^{(t)}$ of the E-field must satisfy the continuity condition at the interface, as do the tangential components $H_y^{(i)}, H_y^{(r)}, H_y^{(t)}$ of the H-field. Therefore,

$\boldsymbol{E}_\parallel$ continuity: $E_0\cos\theta + \rho_p E_0\cos\theta = \tau_p E_0\cos\theta \quad\rightarrow\quad 1 + \rho_p = \tau_p.$

$\boldsymbol{H}_\parallel$ continuity: $Z_0^{-1}E_0 - Z_0^{-1}\rho_p E_0 = \dfrac{\tau_p\varepsilon(\omega)E_0\cos\theta}{Z_0\sqrt{\varepsilon(\omega)-\sin^2\theta}} \quad\rightarrow\quad 1 - \rho_p = \dfrac{\tau_p\varepsilon(\omega)\cos\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}}.$

Solving the above equations, we find $\rho_p = \dfrac{\sqrt{\varepsilon(\omega)-\sin^2\theta}-\varepsilon(\omega)\cos\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}+\varepsilon(\omega)\cos\theta}$ and $\tau_p = \dfrac{2\sqrt{\varepsilon(\omega)-\sin^2\theta}}{\sqrt{\varepsilon(\omega)-\sin^2\theta}+\varepsilon(\omega)\cos\theta}$.

e) Since $\varepsilon(\omega)$ is real-valued and negative, ρ_p may be written as follows:

$$\rho_p = \frac{i\sqrt{|\varepsilon(\omega)|+\sin^2\theta}+|\varepsilon(\omega)|\cos\theta}{i\sqrt{|\varepsilon(\omega)|+\sin^2\theta}-|\varepsilon(\omega)|\cos\theta}$$

Thus ρ_p is seen to be the ratio of a complex number to its conjugate, which has a magnitude of 1. Since $|\rho_p| = 1$, the reflectivity is 100%. This does not contradict the existence of electromagnetic waves within the plasma, because the time-averaged Poynting vector of the plane-wave inside the plasma, like that of an evanescent wave, has a vanishing z-component.