Spring 2013 Opti 501

Solutions to Written Comprehensive Exam Problems

Solution to Problem 1) a) The E-field energy-density is $\frac{1}{2}\varepsilon_o|\boldsymbol{E}|^2$. Since the E-field oscillates with frequency ω_o, time-averaging yields the average E-field energy-density as $\frac{1}{4}\varepsilon_o E_0^2$. Multiplying this into the volume cTA of the pulse, we obtain the E-field energy of the pulse as $\frac{1}{4}\varepsilon_o cTAE_0^2$. Similarly, the amplitude of the H-field of the light is $H_0 = E_0/Z_o$. Since the time-averaged magnetic energy density in vacuum is given by $\frac{1}{4}\mu_o H_0^2 = \frac{1}{4}\varepsilon_o E_0^2$, the magnetic energy of the pulse is equal to its electric energy. The total energy is thus given by $\frac{1}{2}\varepsilon_o cTAE_0^2$.

Alternatively, we may compute the time-averaged Poynting vector as follows:

$$<\boldsymbol{S}> = \tfrac{1}{2}\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \tfrac{1}{2}E_0 H_0\hat{\boldsymbol{z}} = [E_0^2/(2Z_o)]\hat{\boldsymbol{z}}.$$

This is the rate of flow of energy per unit area per unit time at any given cross-section of the light pulse. Multiplication with A and T then yields the total energy of the pulse as $ATE_0^2/(2Z_o)$. Considering that $\varepsilon_o c = 1/Z_o$, the two expressions obtained above for the total pulse energy are exactly the same.

b) The reflected pulse has the same frequency ω_o and the same wavelength $\lambda_o = 2\pi c/\omega_o$ as the incident pulse. Its polarization state is also linear and in the same direction as the incident polarization. The pulse duration and cross-sectional area remain T and A, respectively. The only things that change are the field amplitudes E_o and H_o, which are multiplied by the Fresnel reflection coefficient $\rho = (1-n)/(1+n)$. The reflected pulse energy is therefore given by ρ^2 times the incident pulse energy, that is, $(1-n)^2 ATE_0^2/[2Z_o(1+n)^2]$.

c) The Fresnel transmission coefficient at the entrance facet of the glass slab is $\tau = 1+\rho = 2/(1+n)$. This means that the E-field amplitude inside the glass slab is $2E_0/(1+n)$. The H-field amplitude is n times the E-field amplitude divided by Z_o, that is, $H_0 = 2nE_0/[Z_o(1+n)]$. Therefore, the z-component of the Poynting vector inside the slab is $<S_z> = 2nE_0^2/[Z_o(1+n)^2]$. Since the pulse duration T and the cross-sectional area A inside the slab remain the same as outside, the total energy of the transmitted pulse is $2nATE_0^2/[Z_o(1+n)^2]$. Other properties of the transmitted pulse are: frequency $= \omega_o$, wavelength $\lambda = \lambda_o/n$, pulse length $= cT/n$, polarization state = linear and in the same direction as the incident pulse.

Alternatively, one may evaluate the energy densities of the E and H fields separately, then add them together. We find

$$\text{Time-averaged } E\text{-field energy density} = \tfrac{1}{4}\varepsilon_o\varepsilon(\tau E_0)^2 = \tfrac{1}{4}\varepsilon_o n^2[2E_0/(1+n)]^2 = \varepsilon_o n^2 E_0^2/(1+n)^2.$$

$$\text{Time-averaged } H\text{-field energy density} = \tfrac{1}{4}\mu_o(n\tau E_0/Z_o)^2 = \tfrac{1}{4}\mu_o\{2nE_0/[Z_o(1+n)]\}^2 = \varepsilon_o n^2 E_0^2/(1+n)^2.$$

Adding the above energy densities, then multiplying by the pulse volume cAT/n yields the same result as before, namely, transmitted pulse energy $= 2\varepsilon_o c\,nATE_0^2/(1+n)^2$.

d) Reflected plus transmitted pulse energy =

$$(1-n)^2ATE_0^2/[2Z_o(1+n)^2] + 2nATE_0^2/[Z_o(1+n)^2] = ATE_0^2/(2Z_o).$$

This is the same as the incident pulse energy; therefore, energy is conserved.

Solution to Problem 2) a) There are four sources in classical electrodynamics that are generally associated with material media and described as continuous functions of the space-time coordinates, $\boldsymbol{r}$ and t. These are: (i) free charge density $\rho_{\text{free}}(\boldsymbol{r},t)$, (ii) free current density $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$, (iii) polarization $\boldsymbol{P}(\boldsymbol{r},t)$, and magnetization $\boldsymbol{M}(\boldsymbol{r},t)$. While ρ_{free} is a scalar entity, the other three sources are vectorial in nature. $\rho_{\text{free}}(\boldsymbol{r},t)$ is the electrical charge (e.g., that of electrons and protons) per unit volume at a given point in space-time. $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$ is the electrical current density (i.e., charge crossing unit area per unit time) produced by the motion of free charges. $\boldsymbol{P}(\boldsymbol{r},t)$ is the density of electric dipoles (i.e., dipole moment per unit volume), and $\boldsymbol{M}(\boldsymbol{r},t)$ is the density of magnetic dipoles at a given point in space-time.

b) There are four fields in the classical theory: (i) electric field $\boldsymbol{E}(\boldsymbol{r},t)$, (ii) electric displacement $\boldsymbol{D}(\boldsymbol{r},t)$, (iii) magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$, and (iv) magnetic induction $\boldsymbol{B}(\boldsymbol{r},t)$. The fields are generally described as continuous functions of the space-time coordinates. $\boldsymbol{E}$ and $\boldsymbol{H}$ may be thought of as pure fields, lacking some of the characteristics that one normally associates with material media, such as mass. In contrast, $\boldsymbol{D}$ and $\boldsymbol{B}$ are composite fields defined by the identities $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_o\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t) = \mu_o\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)$, where ε_o and μ_o are the permittivity and permeability of free space, respectively. All four fields are vectorial in nature.

c) Starting with Maxwell's 2nd equation, one applies the divergence operator to both sides of the equation to arrive at $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) = \boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}} + \boldsymbol{\nabla}\cdot(\partial\boldsymbol{D}/\partial t)$. Since the divergence of the curl of any vector field is always equal to zero, the left-hand-side of the above equation may be set to zero. On the right-hand side, the divergence operator goes inside the time-derivative operator to yield $\partial(\boldsymbol{\nabla}\cdot\boldsymbol{D})/\partial t$. From Maxwell's 1st equation we have $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}$. Substitution into the preceding equation then yields $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}} + (\partial\rho_{\text{free}}/\partial t) = 0$, which is the sought after continuity equation.

d) The polarization and magnetization appearing in Maxwell's equations may be replaced by equivalent bound-charge and bound-current densities. In Maxwell's 1st equation, substituting $\varepsilon_0\boldsymbol{E} + \boldsymbol{P}$ for $\boldsymbol{D}$ and moving $\boldsymbol{\nabla}\cdot\boldsymbol{P}$ to the right-hand-side yields $\varepsilon_o\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho_{\text{free}} - \boldsymbol{\nabla}\cdot\boldsymbol{P}$. This indicates that the bound electric charge density associated with $\boldsymbol{P}$ is $\rho_{\text{bound}}^{(e)} = -\boldsymbol{\nabla}\cdot\boldsymbol{P}$. Similarly, one can substitute for $\boldsymbol{H}$ and $\boldsymbol{D}$ in Maxwell's 2nd equation in terms of $\boldsymbol{B}$ and $\boldsymbol{E}$ to arrive at $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_o[\boldsymbol{J}_{\text{free}} + (\partial\boldsymbol{P}/\partial t) + \mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}] + \mu_o\varepsilon_o(\partial\boldsymbol{E}/\partial t)$. The terms bundled together with $\boldsymbol{J}_{\text{free}}$ on the right-hand-side of the preceding equation then represent the bound electric current density $\boldsymbol{J}_{\text{bound}}^{(e)}$ associated with polarization $(\partial\boldsymbol{P}/\partial t)$, and with magnetization $(\mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{M})$.

Alternatively, one may leave the 1st and 2nd equations intact, and modify the 3rd and 4th equations of Maxwell by substituting for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of $\boldsymbol{D}$ and $\boldsymbol{H}$. One will find

$$\boldsymbol{\nabla}\times\boldsymbol{D} = -\varepsilon_o[(\partial\boldsymbol{M}/\partial t) - \varepsilon_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}] - \mu_o\varepsilon_o(\partial\boldsymbol{H}/\partial t),$$

$$\mu_o \boldsymbol{\nabla} \cdot \boldsymbol{H} = -\boldsymbol{\nabla} \cdot \boldsymbol{M}.$$

The above equations reveal that the magnetization $\boldsymbol{M}$ may be replaced by a bound magnetic charge-density $\rho_{\text{bound}}^{(\text{m})} = -\boldsymbol{\nabla} \cdot \boldsymbol{M}$ and a bound magnetic current-density $\boldsymbol{J}_{\text{bound}}^{(\text{m})} = \partial \boldsymbol{M}/\partial t$. Similarly, the polarization $\boldsymbol{P}$ may be replaced by $\boldsymbol{J}_{\text{bound}}^{(\text{m})} = -\varepsilon_o^{-1} \boldsymbol{\nabla} \times \boldsymbol{P}$.

e) In the case of electric bound charge and current densities we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{e})} = \boldsymbol{\nabla} \cdot (\partial \boldsymbol{P}/\partial t) + \mu_o^{-1} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{M}) = \partial(\boldsymbol{\nabla} \cdot \boldsymbol{P})/\partial t = -\partial \rho_{\text{bound}}^{(\text{e})}/\partial t \quad \rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{e})} + \partial \rho_{\text{bound}}^{(\text{e})}/\partial t = 0.$$

In the case of magnetic bound charge and current densities we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} = \boldsymbol{\nabla} \cdot (\partial \boldsymbol{M}/\partial t) - \varepsilon_o^{-1} \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{P}) = \partial(\boldsymbol{\nabla} \cdot \boldsymbol{M})/\partial t = -\partial \rho_{\text{bound}}^{(\text{m})}/\partial t \quad \rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} + \partial \rho_{\text{bound}}^{(\text{m})}/\partial t = 0.$$

Clearly, both systems of bound charge and current satisfy the charge-current continuity equation.
