Opti 501 Prelim. Spring 2012

Solution to Problem 1)

a)
$$\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=\rho_{\text{free}}(\boldsymbol{r},t),$$

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\frac{\partial\boldsymbol{D}(\boldsymbol{r},t)}{\partial t},$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0.$$

In the above equations, $\boldsymbol{r}=x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}$ is an arbitrary point in space, while t is an arbitrary instant in time. $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{D}$ is the displacement, and $\boldsymbol{B}$ is the magnetic induction. The fields are related to each other, to the permittivity and permeability of free space, ε_o and μ_o, and to polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$ as follows

$$\boldsymbol{D}(\boldsymbol{r},t)=\varepsilon_o\boldsymbol{E}(\boldsymbol{r},t)+\boldsymbol{P}(\boldsymbol{r},t),$$

$$\boldsymbol{B}(\boldsymbol{r},t)=\mu_o\boldsymbol{H}(\boldsymbol{r},t)+\boldsymbol{M}(\boldsymbol{r},t).$$

The sources of the electromagnetic fields (namely, $\boldsymbol{E}$ and $\boldsymbol{H}$) are the free charge density ρ_{free}, free current density $\boldsymbol{J}_{\text{free}}$, polarization $\boldsymbol{P}$ (which is the density of electric dipole moments), and magnetization $\boldsymbol{M}$ (which is the density of magnetic dipole moments). The operator $\partial/\partial t$ represents partial differentiation with respect to time, $\boldsymbol{\nabla}\cdot$ is the divergence operator, and $\boldsymbol{\nabla}\times$ is the curl operator. The divergence of a vector field such as $\boldsymbol{D}(\boldsymbol{r},t)$, which turns out to be a scalar field, is defined as the integral of $\boldsymbol{D}(\boldsymbol{r},t)$ over a small closed surface, normalized by the enclosed volume. The curl of a vector field such as $\boldsymbol{E}(\boldsymbol{r},t)$, which turns out to be another vector field, when projected onto the surface normal of a small surface element, yields the line integral of $\boldsymbol{E}(\boldsymbol{r},t)$ around the boundary of the small surface element, normalized by the surface area of the element.

b) To derive the charge-current continuity equation from Maxwell's equations, apply the divergence operator to both sides of the second (Maxwell-Ampere) equation. The divergence of curl is always equal to zero and, therefore, the left-hand-side of the equation becomes $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H})=0$. The right-hand side, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}+\partial(\boldsymbol{\nabla}\cdot\boldsymbol{D})/\partial t$, thus becomes zero. Maxwell's first equation (Gauss's law) now allows one to replace $\boldsymbol{\nabla}\cdot\boldsymbol{D}$ with ρ_{free}, yielding the continuity equation as $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}+\partial\rho_{\text{free}}/\partial t=0$. This equation informs that the integrated free current over any closed surface is precisely balanced by changes in the electrical charge contained within the closed surface. If there is a net outflow of the current, the charge within the closed surface must be decreasing, and if there is a net inflow of current, the charge within must be increasing.

c) In the first of Maxwell's equations, we substitute $\boldsymbol{D}=\varepsilon_o\boldsymbol{E}+\boldsymbol{P}$ and obtain

$$\boldsymbol{\nabla}\cdot(\varepsilon_o\boldsymbol{E}+\boldsymbol{P})=\rho_{\text{free}}\quad\rightarrow\quad\varepsilon_o\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho_{\text{free}}-\boldsymbol{\nabla}\cdot\boldsymbol{P}\quad\rightarrow\quad\varepsilon_o\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho_{\text{free}}+\rho_{\text{bound}}^{(e)}.$$

The bound-charge density is thus seen to be $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t).$

In the second Maxwell equation, we multiply both sides by μ_o, then add $\nabla \times \boldsymbol{M}$ to both sides, in order to replace $\boldsymbol{H}$ with $\boldsymbol{B}$ through the identity $\boldsymbol{B} = \mu_o \boldsymbol{H} + \boldsymbol{M}$. We also use $\boldsymbol{D} = \varepsilon_o \boldsymbol{E} + \boldsymbol{P}$ on the right-hand side of the equation to get rid of $\boldsymbol{D}$. We will have

$$\mu_o \nabla \times \boldsymbol{H} + \nabla \times \boldsymbol{M} = \mu_o \boldsymbol{J}_{\text{free}} + \mu_o \frac{\partial(\varepsilon_o \boldsymbol{E} + \boldsymbol{P})}{\partial t} + \nabla \times \boldsymbol{M}$$

$$\rightarrow \qquad \nabla \times \boldsymbol{B} = \mu_o (\boldsymbol{J}_{\text{free}} + \partial \boldsymbol{P}/\partial t + \mu_o^{-1} \nabla \times \boldsymbol{M}) + \mu_o \varepsilon_o \partial \boldsymbol{E}/\partial t$$

$$\rightarrow \qquad \nabla \times \boldsymbol{B} = \mu_o (\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(e)}) + \mu_o \varepsilon_o \partial \boldsymbol{E}/\partial t.$$

The bound electric current density is thus found to be $\boldsymbol{J}_{\text{bound}}^{(e)} = \partial \boldsymbol{P}/\partial t + \mu_o^{-1} \nabla \times \boldsymbol{M}$. Since the remaining Maxwell equations do not contain $\boldsymbol{D}$ and $\boldsymbol{H}$, they remain unchanged.

d) The divergence of $\boldsymbol{J}_{\text{bound}}^{(e)}$ is readily obtained as follows:

$$\nabla \cdot \boldsymbol{J}_{\text{bound}}^{(e)} = \partial(\nabla \cdot \boldsymbol{P})/\partial t + \mu_o^{-1} \nabla \cdot (\nabla \times \boldsymbol{M}).$$

On the right-hand side of the above equation, the divergence of curl is always zero. Also the divergence of $\boldsymbol{P}(\boldsymbol{r}, t)$ is, by definition, $-\rho_{\text{bound}}^{(e)}$. Therefore, $\nabla \cdot \boldsymbol{J}_{\text{bound}}^{(e)} + \partial \rho_{\text{bound}}^{(e)}/\partial t = 0$. This is the charge-current continuity equation for the bound electrical charge and current defined in part (c).

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Solution to Problem 2)

a) $$\boldsymbol{E}_1(\boldsymbol{r},t) = E_o \cos[n(\omega_1)(\omega_1/c)z - \omega_1 t]\hat{\boldsymbol{x}}, \tag{1a}$$

$$\boldsymbol{H}_1(\boldsymbol{r},t) = n(\omega_1) Z_o^{-1} E_o \cos[n(\omega_1)(\omega_1/c)z - \omega_1 t]\hat{\boldsymbol{y}}. \tag{1b}$$

Similarly,

$$\boldsymbol{E}_2(\boldsymbol{r},t) = E_o \cos[n(\omega_2)(\omega_2/c)z - \omega_2 t]\hat{\boldsymbol{x}}, \tag{2a}$$

$$\boldsymbol{H}_2(\boldsymbol{r},t) = n(\omega_2) Z_o^{-1} E_o \cos[n(\omega_2)(\omega_2/c)z - \omega_2 t]\hat{\boldsymbol{y}}. \tag{2b}$$

Here $c = 1/\sqrt{\mu_o \varepsilon_o}$ is the speed of light in vacuum, while $Z_o = \sqrt{\mu_o/\varepsilon_o}$ is the impedance of free space.

b) The rate of flow of electromagnetic (EM) energy is given by the Poynting vector, as follows:

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{r},t) &= \boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t) = E_o\{\cos[n(\omega_1)(\omega_1/c)z - \omega_1 t] + \cos[n(\omega_2)(\omega_2/c)z - \omega_2 t]\}\,\hat{\boldsymbol{x}} \\ &\quad \times Z_o^{-1} E_o\{n(\omega_1)\cos[n(\omega_1)(\omega_1/c)z - \omega_1 t] + n(\omega_2)\cos[n(\omega_2)(\omega_2/c)z - \omega_2 t]\}\,\hat{\boldsymbol{y}} \\ &= Z_o^{-1} E_o^2\{n(\omega_1)\cos^2[n(\omega_1)(\omega_1/c)z - \omega_1 t] + n(\omega_2)\cos^2[n(\omega_2)(\omega_2/c)z - \omega_2 t] \\ &\quad + [n(\omega_1) + n(\omega_2)]\cos[n(\omega_1)(\omega_1/c)z - \omega_1 t]\cos[n(\omega_2)(\omega_2/c)z - \omega_2 t]\}\hat{\boldsymbol{z}} \\ &= \tfrac{1}{2} Z_o^{-1} E_o^2\{[n(\omega_1) + n(\omega_2)] + n(\omega_1)\cos[2\omega_1 n(\omega_1)(z/c) - 2\omega_1 t] + n(\omega_2)\cos[2\omega_2 n(\omega_2)(z/c) - 2\omega_2 t] \\ &\quad + [n(\omega_1) + n(\omega_2)]\cos\{[\omega_1 n(\omega_1) + \omega_2 n(\omega_2)](z/c) - (\omega_1 + \omega_2)t\} \\ &\quad + [n(\omega_1) + n(\omega_2)]\cos\{[\omega_1 n(\omega_1) - \omega_2 n(\omega_2)](z/c) - (\omega_1 - \omega_2)t\}\}\,\hat{\boldsymbol{z}}. \end{aligned} \tag{3}$$

c) In the preceding expression, the terms with frequencies $2\omega_1$, $2\omega_2$, and $(\omega_1 + \omega_2)$ are rapidly-oscillating functions of time which quickly average to zero. The first term, however, is a constant, and the last term, which varies slowly with time, co-propagates with the envelope of the beat signal. Dropping the rapidly-oscillating terms, we will have

$$\begin{aligned} S_z(\boldsymbol{r},t) &\approx \tfrac{1}{2}[n(\omega_1) + n(\omega_2)] Z_o^{-1} E_o^2 \{1 + \cos\{[\omega_2 n(\omega_2) - \omega_1 n(\omega_1)](z/c) - (\omega_2 - \omega_1)t\}\} \\ &\approx [n(\omega_1) + n(\omega_2)] Z_o^{-1} E_o^2 \cos^2\left\{\tfrac{1}{2}\Delta\omega\left(\left.\frac{\mathrm{d}[\omega n(\omega)]}{c\,\mathrm{d}\omega}\right|_{\omega_o} z - t\right)\right\}. \end{aligned} \tag{4}$$

In the above equation, the rate-of-flow of the beat signal's EM energy is seen to travel along the z-axis at the constant velocity c/n_g, where $n_g = \mathrm{d}[\omega n(\omega)]/\mathrm{d}\omega|_{\omega=\omega_o}$ is the group refractive index of the medium at the center frequency ω_o of the beat signal. The energy flow-rate is thus seen to propagate along the z-axis at the group velocity $V_g = c/n_g$. Note that the final expression obtained in Eq. (4) is positive everywhere, whereas the rapidly-oscillating terms that were dropped from Eq. (3) keep switching direction (between $+z$ and $-z$) at very high frequencies.
