Problem 1) a) The divergence operator $\nabla\cdot$ acting on the *D*-field means that the *D*-field is integrated over the surface of a small volume element surrounding an arbitrary point $\boldsymbol{r}$ in space at a fixed time t; the integral must subsequently be normalized by the volume of the chosen element to yield the divergence of the *D*-field. According to Maxwell's 1st equation, the result of this operation on the *D*-field is going to be equal to the density of free charge, ρ_{free}, at that point $\boldsymbol{r}$ in space and at that instant t of time. The displacement field $\boldsymbol{D}(\boldsymbol{r},t)$ is related to the permittivity of free space ε_{o}, the local *E*-field $\boldsymbol{E}(\boldsymbol{r},t)$, and the local polarization $\boldsymbol{P}(\boldsymbol{r},t)$ as follows: $\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_{\text{o}}\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)$.

The boundary condition associated with Maxwell's 1st equation states that the discontinuity in the perpendicular component $\boldsymbol{D}_{\perp}$ of the *D*-field at any given surface or interface must be equal to the local surface-charge-density $\sigma_{\text{free}}(\boldsymbol{r},t)$. Thus at a given point $(\boldsymbol{r},t)$ in space-time, $\boldsymbol{D}_{\perp}(\boldsymbol{r}^{+},t)$ immediately above the surface minus $\boldsymbol{D}_{\perp}(\boldsymbol{r}^{-},t)$ immediately below the surface must be equal in magnitude to $\sigma_{\text{free}}(\boldsymbol{r},t)$ at the surface.

b) The curl operator $\nabla\times$ acting on the *H*-field means that an arbitrarily small loop must be chosen around the point $\boldsymbol{r}$ in space, the integral of the *H*-field around the loop evaluated, then normalized by the surface area of the loop. (The value used for the *H*-field at all points around the loop must be obtained at the same instant of time, t.) According to Maxwell's 2nd equation, the result of the above operation will be equal to the sum of two terms:

i) the projection, on the surface-normal of the loop, of the local free-current-density, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$;
ii) the projection, on the surface-normal of the loop, of the time-derivative of the local $\boldsymbol{D}(\boldsymbol{r},t)$.

The direction of the aforementioned surface-normal is chosen in accordance with the right-hand rule, in conjunction with the direction of travel around the loop when evaluating the integral of the *H*-field. The above description of Maxwell's 2nd equation applies to all small loops, irrespective of the shape and/or orientation of the loop.

The boundary condition associated with Maxwell's 2nd equation states that the discontinuity in the tangential component $\boldsymbol{H}_{\parallel}$ of the *H*-field at any given surface or interface must be equal in magnitude and perpendicular in direction to the local surface-current-density $J_{\text{s\_free}}(\boldsymbol{r},t)$. Thus, at a given point $(\boldsymbol{r},t)$ in space-time, $\boldsymbol{H}_{\parallel}(\boldsymbol{r}^{+},t)$ immediately above the surface minus $\boldsymbol{H}_{\parallel}(\boldsymbol{r}^{-},t)$ immediately below the surface must be equal to $\boldsymbol{J}_{\text{s\_free}}(\boldsymbol{r},t)\times\hat{\boldsymbol{n}}$ at the surface, where $\hat{\boldsymbol{n}}$ is the surface-normal at $\boldsymbol{r}$.

c) The curl operation was described in part (b) above. The magnetic induction $\boldsymbol{B}(\boldsymbol{r},t)$ is related to the permeability μ_{o} of free space, the local *H*-field $\boldsymbol{H}(\boldsymbol{r},t)$, and the local magnetization $\boldsymbol{M}(\boldsymbol{r},t)$ through the following relation: $\boldsymbol{B}(\boldsymbol{r},t) = \mu_{\text{o}}\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)$. Thus, according to Maxwell's 3rd equation, the integral of the *E*-field around any small loop surrounding the point $\boldsymbol{r}$ and evaluated at time t, when normalized by the area of the loop, will be equal in magnitude and opposite in direction to the projection on the surface-normal of the loop of the time-derivative of the local *B*-field. The time-derivative of the *B*-field, of course, is the difference between $\boldsymbol{B}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t+\Delta t)$, normalized by Δt, in the limit with Δt is sufficiently small.

The boundary condition associated with Maxwell's 3rd equation states that the tangential component $\boldsymbol{E}_{\parallel}$ of the *E*-field at any given surface or interface must be continuous. Thus, at a given point $(\boldsymbol{r},t)$ in space-time, $\boldsymbol{E}_{\parallel}(\boldsymbol{r}^{+},t)$ immediately above the surface must be equal to $\boldsymbol{E}_{\parallel}(\boldsymbol{r}^{-},t)$ immediately below the surface.

d) According to Maxwell's 4th equation, the divergence of $\boldsymbol{B}(\boldsymbol{r},t)$ is always and everywhere equal to zero, meaning that the integral of $\boldsymbol{B}(\boldsymbol{r},t)$ over the surface enclosing *any* volume of space (large or small) is identically zero, provided that the B-field at all points on the surface is evaluated at the same instant of time, t. Thus, whatever magnetic flux enters the volume, must also leave the volume, ensuring that no sources and/or sinks of the B-field reside within the volume. This is equivalent to saying that no magnetic monopoles exist in Nature.

The boundary condition associated with Maxwell's 4th equation states that no discontinuities exist in the perpendicular component $\boldsymbol{B}_{\perp}$ of the B-field at surfaces and interfaces. Thus, at a given point $(\boldsymbol{r},t)$ in space-time, $\boldsymbol{B}_{\perp}(\boldsymbol{r}^{+},t)$ immediately above the surface is exactly equal to $\boldsymbol{B}_{\perp}(\boldsymbol{r}^{-},t)$ immediately below the surface.

Problem 2) a) The expression for the E-field is $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{\text{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$. The k-vector is, in general, complex-valued, meaning that $\boldsymbol{k}=\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}''$. The propagation direction is given by $\boldsymbol{k}'$, while $\boldsymbol{k}''$ specifies the direction along which the beam is attenuated (whenever $k''\neq 0$). The E-field amplitude is given by the complex-valued vector $\boldsymbol{E}_{\text{o}}=\boldsymbol{E}_{\text{o}}'+\mathrm{i}\boldsymbol{E}_{\text{o}}''$. In the MKSA system of units, $\boldsymbol{E}$ and $\boldsymbol{E}_{\text{o}}$ have units of *volt*/*meter*, $\boldsymbol{k}$ has units of m^{-1}, and ω has units of sec^{-1} (or *radians*/*sec*).

b) If the real-valued vectors $\boldsymbol{E}_{\text{o}}'$ and $\boldsymbol{E}_{\text{o}}''$ are aligned with each other, or if one of them happens to be zero, then the E-field is said to be linearly-polarized. When both $\boldsymbol{E}_{\text{o}}'$ and $\boldsymbol{E}_{\text{o}}''$ are non-zero and also have different orientations in space, the E-field is circularly or elliptically polarized. (For circular polarization, $\boldsymbol{E}_{\text{o}}'$ and $\boldsymbol{E}_{\text{o}}''$ must have equal magnitudes and be perpendicular to each other.)

c) The expression for the H-field is $\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{\text{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$. The H-field amplitude is given by the complex-valued vector $\boldsymbol{H}_{\text{o}}=\boldsymbol{H}_{\text{o}}'+\mathrm{i}\boldsymbol{H}_{\text{o}}''$. In the MKSA system of units, $\boldsymbol{H}$ and $\boldsymbol{H}_{\text{o}}$ have units of *ampere*/*meter*.

d) In the absence of $\boldsymbol{P}(\boldsymbol{r},t)$ and $\rho_{\text{free}}(\boldsymbol{r},t)$, we will have $\boldsymbol{D}(\boldsymbol{r},t)=\varepsilon_{\text{o}}\boldsymbol{E}(\boldsymbol{r},t)$, and Maxwell's 1st equation reduces to $\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t)=0$. Substituting the E-field distribution of part (a) in this equation then yields $\boldsymbol{k}\cdot\boldsymbol{E}_{\text{o}}=0$, which is the constraint imposed on $\boldsymbol{k}$ and $\boldsymbol{E}_{\text{o}}$ by Maxwell's 1st equation.

e) Using the E- and H-field distributions given in (a) and (c), Maxwell's 2nd equation yields: $\boldsymbol{k}\times\boldsymbol{H}_{\text{o}}=-\omega\varepsilon_{\text{o}}\boldsymbol{E}_{\text{o}}$, which is the only constraint imposed by the 2nd equation on $\boldsymbol{k}$, ω, $\boldsymbol{E}_{\text{o}}$, and $\boldsymbol{H}_{\text{o}}$.

f) Using the E- and H-field distributions given in (a) and (c), Maxwell's 3rd equation yields: $\boldsymbol{k}\times\boldsymbol{E}_{\text{o}}=\omega\mu_{\text{o}}\boldsymbol{H}_{\text{o}}$, which is the only constraint imposed by the 3rd equation on $\boldsymbol{k}$, ω, $\boldsymbol{E}_{\text{o}}$, and $\boldsymbol{H}_{\text{o}}$.

g) In the absence of $\boldsymbol{M}(\boldsymbol{r},t)$ we will have $\boldsymbol{B}(\boldsymbol{r},t)=\mu_o\boldsymbol{H}(\boldsymbol{r},t)$, and Maxwell's 4th equation reduces to $\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r},t)=0$. Substituting the H-field distribution of part (c) in this equation then yields $\boldsymbol{k}\cdot\boldsymbol{H}_o=0$, which is the sole constraint imposed on $\boldsymbol{k}$ and $\boldsymbol{H}_o$ by Maxwell's 4th equation.

h) In part (f) we found $\boldsymbol{H}_o=(\omega\mu_o)^{-1}\boldsymbol{k}\times\boldsymbol{E}_o$. Substituting this expression for $\boldsymbol{H}_o$ into the constraint obtained in part (e) yields: $\boldsymbol{k}\times[(\omega\mu_o)^{-1}\boldsymbol{k}\times\boldsymbol{E}_o]=-\omega\varepsilon_o\boldsymbol{E}_o$. Using the vector identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$ we write the preceding equation as $(\boldsymbol{k}\cdot\boldsymbol{E}_o)\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}_o=-\mu_o\varepsilon_o\omega^2\boldsymbol{E}_o$. From part (d) we know that $\boldsymbol{k}\cdot\boldsymbol{E}_o=0$; therefore, $(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}_o=\mu_o\varepsilon_o\omega^2\boldsymbol{E}_o$. Dropping $\boldsymbol{E}_o$ from both sides of this equation and using the fact that $\mu_o\varepsilon_o=1/c^2$ now yields $k^2=(\omega/c)^2$, which is the desired dispersion relation.