Opti 501 Solutions to Prelims questions Spring 2010

Problem 1) a) At normal incidence, $\theta = 0$, we find

$$\rho_p = \frac{n_1\sqrt{\epsilon_2} - \epsilon_2}{n_1\sqrt{\epsilon_2} + \epsilon_2} = \frac{n_1 - \sqrt{\epsilon_2}}{n_1 + \sqrt{\epsilon_2}},$$

$$\rho_s = \frac{n_1 - \sqrt{\epsilon_2}}{n_1 + \sqrt{\epsilon_2}}.$$

Thus $\rho_p = \rho_s$ at normal incidence. This is expected, of course, because at normal incidence there is no difference between p- and s-polarization; that is, at $\theta = 0$ the plane of incidence, relative to which p- and s- directions are defined, becomes indeterminate.

b) TIR occurs when the second medium is also transparent, that is, when ϵ_2 is real and positive. In TIR we must have $|\rho_p| = 1$ and $|\rho_s| = 1$. Each expression is the ratio of the sum to the difference of two numbers. When two complex numbers have a phase difference of 90°, their sum and difference will have exactly the same magnitudes and, therefore, the ratio of difference to sum will have a magnitude of 1. In the formulas for ρ_p and ρ_s, since n_1, ϵ_2, $\sin\theta$, and $\cos\theta$ are all real numbers, the only way that the numbers in the numerator and denominator of ρ_p and ρ_s could acquire a 90° phase difference would be for the square-roots to become purely imaginary. Therefore, when $\epsilon_2 - n_1^2 \sin^2\theta < 0$, the square-roots become imaginary, the two numbers appearing in the numerator and denominator of each expression for the Fresnel coefficient become 90° apart, and we obtain $|\rho_p| = |\rho_s| = 1$. The condition for TIR is, therefore,

$$\epsilon_2 - n_1^2 \sin^2\theta < 0 \Rightarrow \sin^2\theta > \frac{\epsilon_2}{n_1^2} = \frac{n_2^2}{n_1^2}$$

$$\Rightarrow \sin\theta > \frac{n_2}{n_1}$$

Since $\sin\theta$ can't exceed unity, we must have $n_2 < n_1$. The critical angle $\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$.

From the above argument it is clear that the same critical angle pertains to both p- and s-polarized plane-waves. However, one can argue on physical grounds that the critical angles must be the same, as follows: For $|\rho_p|=1$ and/or $|\rho_s|=1$, we must have 100% of the incident optical energy reflected back into the incidence medium. This means that the transmitted beam in both cases (i.e., p- and s-polarization) can't carry any energy and must, therefore, be evanescent. Now, the condition for evanescence is a condition on the k-vector of the transmitted plane-wave, not on its E- and H-field amplitudes. When the boundary conditions are matched, the k-vector is determined by the requirement that k_x and k_y (and also ω) be the same for the incident, reflected, and transmitted plane-waves. This step in matching the boundary condition is what we've called "Snell's law." Thus, transition from propagating to evanescent field occurs as a result of the application of Snell's law to the k-vectors, which is independent of the incident beam being p- or s-polarized. The condition for TIR, therefore, is not dependent on the state of polarization of the incident beam.

c) At Brewster's angle $\rho_p=0 \Rightarrow n_1\sqrt{\epsilon_2-n_1^2\sin^2\theta}=\epsilon_2\cos\theta \Rightarrow$

$$n_1^2(\epsilon_2-n_1^2\sin^2\theta)=\epsilon_2^2\cos^2\theta \Rightarrow \frac{n_1^2\epsilon_2}{\epsilon_2^2\cos^2\theta}-\frac{n_1^4\sin^2\theta}{\epsilon_2^2\cos^2\theta}=1 \Rightarrow \left(\frac{n_1^2}{\epsilon_2}\right)(1+\tan^2\theta)-\left(\frac{n_1^2}{\epsilon_2}\right)^2\tan^2\theta=1$$

$$\Rightarrow \cancel{\frac{n_1^2}{\epsilon_2}-1}=\left(\frac{n_1^2}{\epsilon_2}\right)\cancel{\left(\frac{n_1^2}{\epsilon_2}-1\right)}\tan^2\theta \Rightarrow \tan^2\theta=\frac{\epsilon_2}{n_1^2} \Rightarrow \tan\theta=\frac{n_2+ik_2}{n_1}$$

(In the above derivation, we have ignored the trivial solution $\frac{n_1^2}{\epsilon_2}-1=0$, which yields $n_1^2=\epsilon_2 \Rightarrow n_1=\sqrt{\epsilon_2} \Rightarrow$ no mismatch of impedances at the interface.)

Now, for a non-trivial solution to exist, we must have $\tan\theta=(n_2/n_1)+i(k_2/n_1)$. However, $\tan\theta$ is real-valued; therefore k_2 must be zero. Moreover, $0\le\theta<90°$ requires that $\tan\theta>0$. Therefore $n_2>0$. Thus for Brewster's angle to exist we must have ϵ_2 real-valued and positive. The incidence angle at which $\rho_p=0$ is then given by $\theta=\tan^{-1}(\sqrt{\epsilon_2}/n_1)$. Note that $n_2=\sqrt{\epsilon_2}>n_1$ and $n_2<n_1$ are both possible.

Brewster's angle does not exist for s-light because:

$\rho_s = 0 \Rightarrow n_1 \cos\theta = \sqrt{\epsilon_2 - n_1^2 \sin^2\theta} \Rightarrow n_1^2 \cos^2\theta = \epsilon_2 - n_1^2 \sin^2\theta \Rightarrow n_1^2(\sin^2\theta + \cos^2\theta) = \epsilon_2$

$\Rightarrow n_1^2 = \epsilon_2$, i.e., trivial case when there is no impedance mis-match at the interface between the two media.

d) We already saw in part (c) above that for ρ_p to become zero, we must have $K_2 = 0$. Thus any absorption (or gain for that matter) is incompatible with the existence of a Brewster's angle.

e) When $\theta \to 90°$, $\sin\theta \to 1$ and $\cos\theta \to 0$. Therefore,

$$\rho_p \to \frac{n_1\sqrt{\epsilon_2 - n_1^2}}{n_1\sqrt{\epsilon_2 - n_1^2}} = +1 \qquad \rho_s \to \frac{-\sqrt{\epsilon_2 - n_1^2}}{+\sqrt{\epsilon_2 - n_1^2}} = -1$$

With s-light, the incident and reflected plane-waves cancel each other out, and there ~~will~~ be no E-field parallel to the surface, nor an H-field ⊥ to the surface. The same thing happens with p-light; however, one must be careful in interpreting the meaning of ρ_p, as it is the ratio of E^r_{x0} to E^i_{x0}. Indeed, the ratio of E^r_{z0} to E^i_{z0} approaches -1 for p-light as well, and so does H^r_{y0}/H^i_{y0}.

f) The answer is yes, there can be total reflection at the interface when $K_2 \neq 0$ but then we must have $n_2 = 0$; in other words, we must have ϵ_2 real-valued and negative. The proof is as follows:

i) Case of s-polarization.

$$|\rho_s|^2 = 1 \Rightarrow (n_1\cos\theta - \sqrt{\epsilon_2 - n_1^2\sin^2\theta})(n_1\cos\theta - \sqrt{\epsilon_2 - n_1^2\sin^2\theta})^* = (n_1\cos\theta + \sqrt{\epsilon_2 - n_1^2\sin^2\theta})(n_1\cos\theta + \sqrt{\epsilon_2 - n_1^2\sin^2\theta})^*$$

$$\Rightarrow -n_1\cos\theta\sqrt{\epsilon_2 - n_1^2\sin^2\theta}^{\,*} - n_1\cos\theta\sqrt{\epsilon_2 - n_1^2\sin^2\theta} = n_1\cos\theta\sqrt{\epsilon_2 - n_1^2\sin^2\theta}^{\,*} + n_1\cos\theta\sqrt{\epsilon_2 - n_1^2\sin^2\theta}$$

$$\Rightarrow \sqrt{\epsilon_2 - n_1^2\sin^2\theta} + \sqrt{\epsilon_2 - n_1^2\sin^2\theta}^{\,*} = 0 \Rightarrow \sqrt{\epsilon_2 - n_1^2\sin^2\theta} = \text{purely imaginary} = i\alpha$$

arbitrary real number

$$\Rightarrow \epsilon_2 - n_1^2\sin^2\theta = -\alpha^2 \Rightarrow \epsilon_2 = n_1^2\sin^2\theta - \alpha^2 \Rightarrow \epsilon_2 \text{ is purely real.}$$

real number

Now, if ϵ_2 is a positive real number, we'll have the condition for conventional TIR at the interface between two transparent media, namely, $\sin\theta > \frac{n_2}{n_1} = \frac{\sqrt{\epsilon_2}}{n_1}$.
However, if ϵ_2 is a negative real number, then $\sqrt{\epsilon_2 - n_1^2\sin^2\theta}$ will be ~~negative~~ imaginary for all values of θ and, therefore, 100% reflection occurs at all θ.

ii) Case of p-polarization:

$$|\rho_p|^2 = 1 \Rightarrow (n_1\sqrt{\epsilon_2 - n_1^2\sin^2\theta} - \epsilon_2\cos\theta)(n_1\sqrt{\epsilon_2 - n_1^2\sin^2\theta} - \epsilon_2\cos\theta)^* = (n_1\sqrt{\epsilon_2 - n_1^2\sin^2\theta} + \epsilon_2\cos\theta)(n_1\sqrt{\epsilon_2 - n_1^2\sin^2\theta} + \epsilon_2\cos\theta)^*$$

$$\Rightarrow -\cancel{n_1}\sqrt{\epsilon_2 - n_1^2\sin^2\theta}\,\epsilon_2^*\cancel{\cos\theta} - \epsilon_2\cancel{\cos\theta}\,\cancel{n_1}\sqrt{\epsilon_2 - n_1^2\sin^2\theta}^* = \cancel{n_1}\sqrt{\epsilon_2 - n_1^2\sin^2\theta}\,\epsilon_2^*\cancel{\cos\theta} + \epsilon_2\cancel{\cos\theta}\,\cancel{n_1}\sqrt{\epsilon_2 - n_1^2\sin^2\theta}^*$$

$$\Rightarrow \epsilon_2^*\sqrt{\epsilon_2 - n_1^2\sin^2\theta} + \epsilon_2\sqrt{\epsilon_2 - n_1^2\sin^2\theta}^* = 0 \Rightarrow \epsilon_2^*\sqrt{\epsilon_2 - n_1^2\sin^2\theta} = \text{Purely imaginary} = i\alpha$$

arbitrary real number

$$\Rightarrow (\epsilon_2' - i\epsilon_2'')^2(\epsilon_2' - n_1^2\sin^2\theta + i\epsilon_2'') = -\alpha^2$$

arbitrary Real number

$$\Rightarrow (\epsilon_2'^2 - \epsilon_2''^2 - 2i\epsilon_2'\epsilon_2'')(\epsilon_2' - n_1^2\sin^2\theta + i\epsilon_2'') = -\alpha^2 \Rightarrow \begin{cases} (\epsilon_2'^2 - \epsilon_2''^2)(\epsilon_2' - n_1^2\sin^2\theta) + 2\epsilon_2'\epsilon_2''^2 = -\alpha^2 \\ (\epsilon_2'^2 - \epsilon_2''^2)\epsilon_2'' - 2\epsilon_2'\epsilon_2''(\epsilon_2' - n_1^2\sin^2\theta) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \epsilon_2'^3 - \epsilon_2'^2 n_1^2\sin^2\theta - \epsilon_2''^2(\epsilon_2' - n_1^2\sin^2\theta - 2\epsilon_2') = -\alpha^2 \\ \epsilon_2''(\epsilon_2''^2 + \epsilon_2'^2 - 2\epsilon_2' n_1^2\sin^2\theta) = 0 \Rightarrow \begin{cases} \epsilon_2'' = 0 & \leftarrow \text{first Solution} \\ \epsilon_2''^2 = 2\epsilon_2' n_1^2\sin^2\theta - \epsilon_2'^2 & \leftarrow \text{Second Solution} \end{cases} \end{cases}$$

First Solution: $\epsilon_2'' = 0 \Rightarrow \epsilon_2'^3 - \epsilon_2'^2 n_1^2\sin^2\theta = -\alpha^2 \Rightarrow \epsilon_2'^2(\epsilon_2' - n_1^2\sin^2\theta) = -\alpha^2$

$$\Rightarrow \epsilon_2' - n_1^2\sin^2\theta = \text{negative real number} \Rightarrow \begin{cases} \epsilon_2' < 0 \\ 0 < \epsilon_2' < n_1^2\sin^2\theta \end{cases}$$

Therefore, there exists two possibilities: (i) Conventional TIR, where ϵ_2 is real and positive, and $\sin^2\theta > \frac{\epsilon_2}{n_1^2} \Rightarrow \sin\theta > n_2/n_1 = \frac{\sqrt{\epsilon_2}}{n_1}$.
(ii) ϵ_2 real and negative, in which case $|\rho_p| = 1$ for <u>all</u> angles of incidence θ.

Second solution: $\epsilon_2''^2 = 2\epsilon_2' n_1^2 \sin^2\theta - \epsilon_2'^2 \Rightarrow \epsilon_2'^3 - \epsilon_2'^2 n_1^2 \sin^2\theta + (2\epsilon_2' n_1^2 \sin^2\theta - \epsilon_2'^2)(\epsilon_2' + n_1^2 \sin^2\theta) = -\alpha^2$

$$\Rightarrow \cancel{\epsilon_2'^3} - \cancel{\epsilon_2'^2 n_1^2 \sin^2\theta} + \cancel{2\epsilon_2'^2 n_1^2 \sin^2\theta} + 2\epsilon_2' n_1^4 \sin^4\theta - \cancel{\epsilon_2'^3} - \cancel{\epsilon_2'^2 n_1^2 \sin^2\theta} = -\alpha^2$$

$\Rightarrow 2\epsilon_2' n_1^4 \sin^4\theta = -\alpha^2 \Rightarrow \epsilon_2'$ must be purely negative.

However, if this happens, we'll have $\epsilon_2''^2 = 2\epsilon_2' n_1^2 \sin^2\theta - \epsilon_2'^2 < 0$, which is unacceptable for $\epsilon_2''^2$, a purely positive number. Therefore, a second solution does not exist.

Problem 2) a) $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow I(t) = \frac{dq(t)}{dt} \Rightarrow \frac{dq}{dt} = I_0 \sin(2\pi f t) \Rightarrow$

$$q(t) = \int_{-\infty}^{t} I_0 \sin(2\pi f t')\,dt' = -\frac{I_0}{2\pi f}\cos(2\pi f t) + \text{Constant}.$$

b) $\vec{p}(t) = q(t)\, d\,\hat{z} = -\frac{I_0 d}{2\pi f}\cos(2\pi f t)\,\hat{z}$ ← Constant (static) dipole has been ignored.

c) In spherical coordinates $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$.

$$\mu_0 \vec{H}(\vec{r},t) = \vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) = \frac{\mu_0 I_0 d}{4\pi}\vec{\nabla}\times\left\{\frac{1}{r}\sin\left[2\pi f\left(t-\frac{r}{c}\right)\right](\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\right\} \Rightarrow$$

$$\vec{H}(\vec{r},t) = \frac{I_0 d}{4\pi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi} = \frac{I_0 d\,\hat{\phi}}{4\pi r}\left\{-\sin\theta\frac{\partial}{\partial r}\sin\left[2\pi f\left(t-\frac{r}{c}\right)\right] - \frac{1}{r}\sin\left[2\pi f\left(t-\frac{r}{c}\right)\right]\frac{\partial \cos\theta}{\partial \theta}\right\}$$

$$\Rightarrow \vec{H}(\vec{r},t) = \frac{I_0 d \sin\theta}{4\pi r}\left\{\frac{2\pi f}{c}\cos\left[2\pi f\left(t-\frac{r}{c}\right)\right] + \frac{1}{r}\sin\left[2\pi f\left(t-r/c\right)\right]\right\}\hat{\phi}$$

) $\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \psi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\theta} - \frac{\partial \vec{A}}{\partial t} \Rightarrow$

$$\vec{E}(\vec{r},t) = -\frac{Z_0 I_0 d}{4\pi}\cos\theta \frac{\partial}{\partial r}\left\{\frac{1}{r}\sin[\cdots] - \frac{\lambda_0}{2\pi r^2}\cos[\cdots]\right\}\hat{r} + \frac{Z_0 I_0 d}{4\pi r}\sin\theta\left\{\frac{1}{r}\sin[\cdots] - \frac{\lambda_0}{2\pi r^2}\cos[\cdots]\right\}\hat{\theta}$$

$$-\frac{\mu_0 I_0 d}{4\pi r}(2\pi f)\cos[\cdots](\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \Rightarrow$$

$$\vec{E}(\vec{r},t) = \frac{Z_0 I_0 d}{4\pi}\left\{\left[\frac{1}{r^2}\sin[2\pi f(t - r/c)] - \frac{\lambda_0}{2\pi r^3}\cos[2\pi f(t-\tfrac{r}{c})]\right](2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) + \frac{2\pi}{\lambda_0 r}\sin\theta\cos[2\pi f(t - r/c)]\,\hat{\theta}\right\}$$

e) In the far field region all terms that decrease as $1/r^2$ and $1/r^3$ can be ignored. We'll have:

$$\begin{cases} \vec{E}(\vec{r},t) \simeq \dfrac{Z_0 I_0 d}{2\lambda_0 r}\sin\theta\cos[2\pi f(t - r/c)]\,\hat{\theta} \\ \vec{H}(\vec{r},t) \simeq \dfrac{I_0 d}{2\lambda_0 r}\sin\theta\cos[2\pi f(t - \tfrac{r}{c})]\,\hat{\phi} \end{cases} \quad \leftarrow \text{Note: } \frac{2\pi f}{c} = \frac{2\pi}{\lambda_0}$$

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) = \frac{Z_0}{4}\left(\frac{I_0 d}{\lambda_0}\right)^2 \frac{\sin^2\theta}{r^2}\cos^2[2\pi f(t - r/c)]\,\hat{r}$$