1) a) $\vec{E}(\vec{r},t)=\vec{E}_0\exp\{ik_0\vec{\sigma}\cdot\vec{r}-i\omega t\}$; $\vec{H}(\vec{r},t)=\vec{H}_0\exp(ik_0\vec{\sigma}\cdot\vec{r}-i\omega t)$.

Here $\vec{E}_0=E_{0x}\hat{x}+E_{0y}\hat{y}+E_{0z}\hat{z}$, $\vec{H}_0=H_{0x}\hat{x}+H_{0y}\hat{y}+H_{0z}\hat{z}$, $\vec{\sigma}=\sigma_x\hat{x}+\sigma_y\hat{y}+\sigma_z\hat{z}$.

All Components of $\vec{E}_0,\vec{H}_0,\vec{\sigma}$ are complex-valued.

b) $\vec{\nabla}\cdot\vec{D}=\rho_{free}\Rightarrow\vec{\nabla}\cdot\vec{D}=0\Rightarrow\epsilon_0\epsilon\,\vec{\nabla}\cdot\vec{E}=0\Rightarrow\vec{\sigma}\cdot\vec{E}_0=0.$

$\vec{\nabla}\cdot\vec{B}=0\Rightarrow\mu_0\mu\,\vec{\nabla}\cdot\vec{H}=0\Rightarrow\vec{\nabla}\cdot\vec{H}=0\Rightarrow\vec{\sigma}\cdot\vec{H}_0=0.$

$\vec{\nabla}\times\vec{H}=\vec{J}_{free}+\frac{\partial\vec{D}}{\partial t}\Rightarrow ik_0\vec{\sigma}\times\vec{H}_0=-i\omega\epsilon_0\epsilon(\omega)\vec{E}_0\Rightarrow\vec{\sigma}\times\vec{H}_0=-\frac{\epsilon(\omega)}{Z_0}\vec{E}_0.$

$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\Rightarrow ik_0\vec{\sigma}\times\vec{E}_0=i\omega\mu_0\mu(\omega)\vec{H}_0\Rightarrow\vec{\sigma}\times\vec{E}_0=Z_0\mu(\omega)\vec{H}_0.$

$\Rightarrow\vec{\sigma}\times\vec{E}_0=Z_0\mu(\omega)\vec{H}_0\Rightarrow-\frac{\cancel{Z_0}}{\epsilon(\omega)}\vec{\sigma}\times(\vec{\sigma}\times\vec{H}_0)=\cancel{Z_0}\mu(\omega)\vec{H}_0\Rightarrow$

$\underbrace{\cancel{(\vec{\sigma}\cdot\vec{H}_0)}}_{0}\vec{\sigma}-(\vec{\sigma}\cdot\vec{\sigma})\vec{H}_0=-\mu(\omega)\epsilon(\omega)\vec{H}_0\Rightarrow\vec{\sigma}\cdot\vec{\sigma}=|\vec{\sigma}|^2=\mu(\omega)\epsilon(\omega)$

c) $n(\omega)=|\vec{\sigma}|=\sqrt{\mu(\omega)\epsilon(\omega)}$.

d) $\sigma_x=\sigma_y=0\Rightarrow\begin{cases}\sigma_xE_{x0}+\sigma_yE_{y0}+\sigma_zE_{z0}=0\Rightarrow E_{z0}=0\\ \sigma_xH_{x0}+\sigma_yH_{y0}+\sigma_zH_{z0}=0\Rightarrow H_{z0}=0\\ \sigma_x^2+\sigma_y^2+\sigma_z^2=\mu(\omega)\epsilon(\omega)\Rightarrow\sigma_z=\sqrt{\mu(\omega)\epsilon(\omega)}\end{cases}$

$\vec{\sigma}\times\vec{E}_0=Z_0\mu(\omega)\vec{H}_0\Rightarrow\sigma_z\hat{z}\times(E_{x0}\hat{x}+E_{y0}\hat{y})=\sigma_zE_{x0}\hat{y}-\sigma_zE_{y0}\hat{x}=Z_0\mu(\omega)(H_{0x}\hat{x}+H_{0y}\hat{y})$

$\Rightarrow\begin{cases}\sqrt{\mu\epsilon}\,E_{x0}=Z_0\mu H_{y0}\\ \sqrt{\mu\epsilon}\,E_{y0}=-Z_0\mu H_{x0}\end{cases}\Rightarrow Z(\omega)=\frac{E_{x0}}{H_{y0}}=-\frac{E_{y0}}{H_{x0}}=Z_0\sqrt{\frac{\mu(\omega)}{\epsilon(\omega)}}.$

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2) a) $\langle\vec{S}\rangle=\frac{1}{2}Re(\vec{E}\times\vec{H}^*)=\frac{1}{2}E_0\hat{x}\times(E_0^*/Z_0)\hat{y}=\frac{|E_0|^2}{2Z_0}\hat{z}.$

Total energy content $=(\pi R^2)\tau\langle S_z\rangle=\pi R^2\tau|E_0|^2/2Z_0$.

Total momentum = momentum density x Volume $=\frac{\langle\vec{S}\rangle}{c^2}\pi R^2c\tau=\frac{1}{2}\epsilon_0\pi R^2\tau|E_0|^2\hat{z}.$

b) The massive mirror acquires <u>twice</u> the momentum of the incident pulse, that is,

$\vec{P}_{mirror} = \epsilon_0 \pi R^2 \tau |E_0|^2 \hat{z}$.

Mirror's kinetic energy $= \frac{1}{2} M_0 V^2 = \frac{M_0^2 V^2}{2M_0} = \frac{P_{mirror}^2}{2M_0} \to 0$ when $M_0 \to \infty$.

If the mass M_0 of the mirror happens to be finite, the reflected light will be Doppler-shifted toward red, so that its reduced energy will account for the kinetic energy of the mirror after the pulse has been reflected.

c) Mechanical momentum of perfect absorber = electromagnetic momentum of the light pulse $= \frac{1}{2} \epsilon_0 \pi R^2 \tau |E_0|^2 \hat{z}$.

Non-relativistic treatment: Absorber's kinetic energy $= \frac{P_{absorber}^2}{2M_0} = \frac{\epsilon_0^2 \pi^2 R^4 \tau^2 |E_0|^4}{8M_0}$.

- The kinetic energy of the absorber, which is always less than the pulse energy, comes from the electromagnetic energy of the light pulse. ✓
- The remaining energy of the light pulse is converted to heat, which raises the temperature of the absorber. ✓

Relativistic treatment: The absorber will end up with mass M and velocity $\vec{V}$. The new rest mass of the absorber will be $M_0' = M\sqrt{1 - V^2/c^2}$.

Conservation of energy: $Mc^2 = M_0 c^2 + \frac{\pi R^2 \tau |E_0|^2}{2Z_0} \Rightarrow M = M_0 + \frac{\pi R^2 \tau \epsilon_0 |E_0|^2}{2c}$.

Conservation of momentum: $\vec{P}_{absorber} = \vec{P}_{pulse} \Rightarrow M\vec{V} = \frac{1}{2} \epsilon_0 \pi R^2 \tau |E_0|^2 \hat{z} \Rightarrow$

$$\vec{V} = \frac{\frac{1}{2} \pi R^2 \tau \epsilon_0 |E_0|^2 \hat{z}}{M_0 + \frac{\pi R^2 \tau \epsilon_0 |E_0|^2}{2c}} \Rightarrow V/c = \left(1 + \frac{2M_0 c}{\pi R^2 \tau \epsilon_0 |E_0|^2}\right)^{-1}$$

Once again, the kinetic energy of the absorber comes from the electromagnetic energy of the pulse. The thermal energy, however, has become a part of the absorber, contributing to its increased mass M_0'.