PhD Qualifying Exam, Fall 2021

Opti 501

Solution to Problem 1)

a) $\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}=1 \quad\rightarrow\quad (\mathrm{i}\sigma_x\hat{\boldsymbol{x}}+\sigma_z\hat{\boldsymbol{z}})\cdot(\mathrm{i}\sigma_x\hat{\boldsymbol{x}}+\sigma_z\hat{\boldsymbol{z}})=-\sigma_x^2+\sigma_z^2=1 \quad\rightarrow\quad \sigma_z^2=1+\sigma_x^2.$

b) $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0 \quad\rightarrow\quad \boldsymbol{\sigma}\cdot\boldsymbol{E}_0=0 \quad\rightarrow\quad \mathrm{i}\sigma_x E_{x0}+\sigma_z E_{z0}=0.$

c) $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \quad\rightarrow\quad \boldsymbol{\sigma}\cdot\boldsymbol{H}_0=0 \quad\rightarrow\quad \mathrm{i}\sigma_x H_{x0}+\sigma_z H_{z0}=0.$

d) $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t \;\rightarrow\; \mathrm{i}k_0\boldsymbol{\sigma}\times\boldsymbol{E}_0=\mathrm{i}\omega\mu_0\boldsymbol{H}_0 \rightarrow (\mathrm{i}\sigma_x\hat{\boldsymbol{x}}+\sigma_z\hat{\boldsymbol{z}})\times(E_{x0}\hat{\boldsymbol{x}}+E_{y0}\hat{\boldsymbol{y}}+E_{z0}\hat{\boldsymbol{z}})=Z_0\boldsymbol{H}_0$

$$\rightarrow\; \mathrm{i}\sigma_x E_{y0}\hat{\boldsymbol{z}}+(\sigma_z E_{x0}-\mathrm{i}\sigma_x E_{z0})\hat{\boldsymbol{y}}-\sigma_z E_{y0}\hat{\boldsymbol{x}}=Z_0\boldsymbol{H}_0$$

$$\rightarrow\; Z_0H_{0x}=-\sigma_z E_{y0};\quad Z_0H_{0y}=\sigma_z E_{x0}-\mathrm{i}\sigma_x E_{z0};\quad Z_0H_{0z}=\mathrm{i}\sigma_x E_{y0}.$$

e) If $E_{z0}=0$, then from (b) we have $E_{x0}=0$, and from (d) we find $H_{y0}=0$.

f) If $H_{z0}=0$, then from (c) we have $H_{x0}=0$, and from (d) we find $E_{y0}=0$.

Solution to Problem 2)

a) $\boldsymbol{E}(z,t)=E_{x0}\cos(k_0z-\omega t+\varphi_0)\,\hat{\boldsymbol{x}}$ and $\boldsymbol{H}(z,t)=(E_{x0}/Z_0)\cos(k_0z-\omega t+\varphi_0)\,\hat{\boldsymbol{y}}$. Note that $\boldsymbol{E}$ and $\boldsymbol{H}$ have the same phase φ_0.

$\boldsymbol{E}'(z,t)=E_{x0}\cos(k_0z+\omega t+\varphi_0')\,\hat{\boldsymbol{x}}$ and $\boldsymbol{H}'(z,t)=-(E_{x0}/Z_0)\cos(k_0z+\omega t+\varphi_0')\,\hat{\boldsymbol{y}}$. Again, $\boldsymbol{E}'$ and $\boldsymbol{H}'$ have the same phase φ_0', although it could differ from φ_0.

Total E-field: $\boldsymbol{E}(z,t)+\boldsymbol{E}'(z,t)=E_{x_0}[\cos(k_0z-\omega t+\varphi_0)+\cos(k_0z+\omega t+\varphi_0')]\hat{\boldsymbol{x}}$

$$=2E_{x0}\cos[k_0z+½(\varphi_0'+\varphi_0)]\cos[\omega t+½(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{x}}.$$

Total H-field: $\boldsymbol{H}(z,t)+\boldsymbol{H}'(z,t)=(E_{x0}/Z_0)[\cos(k_0z-\omega t+\varphi_0)-\cos(k_0z+\omega t+\varphi_0')]\hat{\boldsymbol{y}}$

$$=2(E_{x0}/Z_0)\sin[k_0z+½(\varphi_0'+\varphi_0)]\sin[\omega t+½(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{y}}.$$

Since the total E-field at the mirror surfaces must be zero, we will have

First mirror surface at $z=0$: $\quad 2E_{x0}\cos[½(\varphi_0'+\varphi_0)]\cos[\omega t+½(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{x}}=0.$

Second mirror surface at $z=d$: $\quad 2E_{x0}\cos[k_0d+½(\varphi_0'+\varphi_0)]\cos[\omega t+½(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{x}}=0.$

Consequently,

$$\begin{cases}\cos[½(\varphi_0'+\varphi_0)]=0\\ \cos[k_0d+½(\varphi_0'+\varphi_0)]=0\end{cases} \rightarrow \begin{cases}\varphi_0+\varphi_0'=(2n+1)\pi & \text{(odd multiple of }\pi);\\ k_0d=m\pi \;\rightarrow\; d=m\lambda_0/2 & \text{(integer multiple of }\lambda_0/2).\end{cases}$$

The total fields in the cavity are thus found to be

$$\boldsymbol{E}(z,t)=2E_{x0}\sin(k_0z)\cos[\omega t+½(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{x}}.$$

$$\boldsymbol{H}(z,t)=-2(E_{x0}/Z_0)\cos(k_0z)\sin[\omega t+½(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{y}}.$$

b) The surface-current-density equals the total tangential H-field at each mirror surface. For the first mirror at $z = 0$, $\cos(k_0 z) = 1$. For the second mirror at $z = d$, $\cos(k_0 z) = \cos(m\pi) = \pm 1$. The magnitude of the surface-current-density on both mirrors is, therefore, $J_{s0} = 2E_{x0}/Z_0$.

c) The trapped energy per unit cross-sectional area is given by

Trapped E-field energy:

$$\tfrac{1}{2}\varepsilon_0 \int_0^d |\boldsymbol{E}(z,t)|^2 \mathrm{d}z = 2\varepsilon_0 E_{x0}^2 \cos^2[\omega t + \tfrac{1}{2}(\varphi_0' - \varphi_0)] \int_0^d \sin^2(k_0 z)\,\mathrm{d}z$$
$$= \varepsilon_0 E_{x0}^2 d \cos^2[\omega t + \tfrac{1}{2}(\varphi_0' - \varphi_0)].$$

Trapped H-field energy:

$$\tfrac{1}{2}\mu_0 \int_0^d |\boldsymbol{H}(z,t)|^2 \mathrm{d}z = 2\mu_0 (E_{x0}/Z_0)^2 \sin^2[\omega t + \tfrac{1}{2}(\varphi_0' - \varphi_0)] \int_0^d \cos^2(k_0 z)\,\mathrm{d}z$$
$$= \varepsilon_0 E_{x0}^2 d \sin^2[\omega t + \tfrac{1}{2}(\varphi_0' - \varphi_0)].$$

The peak values of E-field and H-field energies (per unit cross-sectional area) are thus equal to $\varepsilon_0 E_{x0}^2 d$. However, there exists a phase difference between these two entities: When the E-field energy is zero, the H-field energy is at a maximum, and vice-versa. At one instant, all the energy is in the E-field; a quarter of a period later, all the energy is in the H-field. The energy thus swings back and forth from one form to the other.

d) $\boldsymbol{S}(z,t) = \boldsymbol{E}(z,t) \times \boldsymbol{H}(z,t)$

$$= -(4E_{x0}^2/Z_0) \sin(k_0 z) \cos(k_0 z) \sin[\omega t + \tfrac{1}{2}(\varphi_0' - \varphi_0)] \cos[\omega t + \tfrac{1}{2}(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{z}}$$
$$= -(E_{x0}^2/Z_0) \sin(2k_0 z) \sin[2\omega t + (\varphi_0' - \varphi_0)]\hat{\boldsymbol{z}}.$$

At the nodes of the E-field, as well as those of the H-field, the Poynting vector is zero. No energy, therefore, crosses these nodes. In between the nodes, the energy flows to the right for one quarter of one oscillation period ($T = 2\pi/\omega$), then flows to the left during the next quarter. The process is then repeated.