Opti 501 Written Comprehensive Exam (Fall 2015)

Solution to Problem 1)

a) In the continuity equation, $\boldsymbol{J}$, a function of the spacetime coordinates $(\boldsymbol{r}, t)$, is the current-density (units: ampere/m^2), while ρ, also a function of $(\boldsymbol{r}, t)$, is the electric charge-density (units: coulomb/m^3). The divergence of current-density, $\boldsymbol{\nabla} \cdot \boldsymbol{J}$, is the normalized rate of outflow of electric charge from a tiny closed surface surrounding the point $\boldsymbol{r}$ at time t; the normalization is by the volume Δv trapped within the closed surface. If the continuity equation is multiplied by Δv, the second term becomes $\partial[\rho(\boldsymbol{r}, t)\Delta v]/\partial t$, which is the time-rate of change of the total electric charge residing within the closed surface. Thus, when the net flux of current out of the surface is positive, the total enclosed charge must decline. The opposite happens when the net flux of the current out of the closed surface is negative.

b) In the absence of $\boldsymbol{P}(\boldsymbol{r}, t)$ and $\boldsymbol{M}(\boldsymbol{r}, t)$, Maxwell's equations are written

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t) = \rho_{\text{free}}(\boldsymbol{r}, t), \tag{1a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) + \varepsilon_0 \partial \boldsymbol{E}(\boldsymbol{r}, t)/\partial t, \tag{1b}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\mu_0 \partial \boldsymbol{H}(\boldsymbol{r}, t)/\partial t, \tag{1c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{H}(\boldsymbol{r}, t) = 0. \tag{1d}$$

Applying the divergence operator to Eq.(1b), we will have

$$\boldsymbol{\nabla} \cdot [\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}, t)] = \boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) + \partial[\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t)]/\partial t. \tag{2}$$

Considering that the divergence of the curl of any vector field is always equal to zero, and that the bracketed term on the right-hand-side of Eq.(2) is the same as the expression appearing on the left-hand-side of Eq.(1a), we may rewrite Eq.(2) as follows:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) + \partial \rho_{\text{free}}(\boldsymbol{r}, t)/\partial t = 0. \tag{3}$$

This, of course, is the charge-current continuity equation for free charges and currents.

c) The bound electric charge-density is the negative of the divergence of polarization, that is,

$$\rho_{\text{bound}}(\boldsymbol{r}, t) = -\boldsymbol{\nabla} \cdot \boldsymbol{P}(\boldsymbol{r}, t). \tag{4}$$

The bound electric current-density consists of two terms, one arising from the time variation of polarization, the other from the curl of magnetization, that is,

$$\boldsymbol{J}_{\text{bound}}(\boldsymbol{r}, t) = \partial \boldsymbol{P}(\boldsymbol{r}, t)/\partial t + \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{M}(\boldsymbol{r}, t). \tag{5}$$

The continuity equation for bound charge and bound current is the same as that for free charge and free current, namely,

$$\boldsymbol{\nabla} \cdot \boldsymbol{J}_{\text{bound}}(\boldsymbol{r}, t) + \partial \rho_{\text{bound}}(\boldsymbol{r}, t)/\partial t = 0. \tag{6}$$

d) In the absence of ρ_{free} and $\boldsymbol{J}_{\text{free}}$, Maxwell's equations are written

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}, t) = 0, \tag{7a}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \partial \boldsymbol{D}(\boldsymbol{r}, t)/\partial t, \tag{7b}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\,\partial \boldsymbol{B}(\boldsymbol{r}, t)/\partial t, \tag{7c}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0. \tag{7d}$$

In the above equations, $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ is the displacement, while $\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}$ is the magnetic induction. Differentiating Eq.(7a) with respect to time, we find

$$\begin{aligned} &\partial \nabla \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P})/\partial t = 0 \quad \rightarrow \quad \partial\,(\nabla \cdot \varepsilon_0 \boldsymbol{E})/\partial t = -\partial\,(\nabla \cdot \boldsymbol{P})/\partial t \\ &\rightarrow \quad \varepsilon_0 \nabla \cdot \partial \boldsymbol{E}(\boldsymbol{r}, t)/\partial t = \partial\, \rho_{\text{bound}}(\boldsymbol{r}, t)/\partial t. \end{aligned} \tag{8}$$

The second of Maxwell's equations may be written in terms of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields as follows:

$$\nabla \times \boldsymbol{B} = \nabla \times \boldsymbol{M} + \mu_0 \partial(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P})/\partial t. \tag{9}$$

Applying the divergence operator to the above equation, we will have

$$\nabla \cdot (\nabla \times \boldsymbol{B}) = \nabla \cdot (\nabla \times \boldsymbol{M} + \mu_0 \partial \boldsymbol{P}/\partial t) + \mu_0 \varepsilon_0 \nabla \cdot (\partial \boldsymbol{E}/\partial t). \tag{10}$$

Considering that the divergence of the curl of any vector field is always equal to zero, and that the last term on the right-hand-side of Eq.(10) may be replaced from Eq.(8), we will have

$$\nabla \cdot (\mu_0^{-1} \nabla \times \boldsymbol{M} + \partial \boldsymbol{P}/\partial t) + \partial \rho_{\text{bound}}/\partial t = 0. \tag{11}$$

This, of course, is the charge-current continuity equation for bound charges and currents.

Solution to Problem 2) In the air, where the refractive index is essentially equal to 1.0, we have $\boldsymbol{k} = k_z\hat{\boldsymbol{z}} = \pm(\omega/c)\hat{\boldsymbol{z}}$, and the E-field to H-field amplitude ratio is $E_0/H_0 = Z_0 = \sqrt{\mu_0/\varepsilon_0}$.

a) Incident beam: $$\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t) = E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[-(\omega/c)z-\omega t]\}, \tag{1a}$$

$$\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t) = -(E_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[-(\omega/c)z-\omega t]\}. \tag{1b}$$

Reflected beam: $$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},t) = \rho E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[(\omega/c)z-\omega t]\}, \tag{2a}$$

$$\boldsymbol{H}^{(\mathrm{r})}(\boldsymbol{r},t) = (\rho E_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[(\omega/c)z-\omega t]\}. \tag{2b}$$

Transmitted beam: $$\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t) = \tau E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[-(\omega/c)z-\omega t]\}, \tag{3a}$$

$$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = -(\tau E_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[-(\omega/c)z-\omega t]\}. \tag{3b}$$

Inside the slab: $$\boldsymbol{E}^{(\mathrm{A})}(\boldsymbol{r},t) = aE_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[-(n\omega/c)z-\omega t]\}, \tag{4a}$$

$$\boldsymbol{H}^{(\mathrm{A})}(\boldsymbol{r},t) = -(anE_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[-(n\omega/c)z-\omega t]\}. \tag{4b}$$

$$\boldsymbol{E}^{(\mathrm{B})}(\boldsymbol{r},t) = bE_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[(n\omega/c)z-\omega t]\}, \tag{5a}$$

$$\boldsymbol{H}^{(\mathrm{B})}(\boldsymbol{r},t) = (bnE_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[(n\omega/c)z-\omega t]\}. \tag{5b}$$

b) At the top of the slab, where $z = 0$, we have

Continuity of $\boldsymbol{E}_\parallel$: $$E_0 + \rho E_0 = aE_0 + bE_0. \tag{6a}$$

Continuity of $\boldsymbol{H}_\parallel$: $$Z_0^{-1}(-E_0 + \rho E_0) = Z_0^{-1}(-anE_0 + bnE_0). \tag{6b}$$

At the bottom of the slab, where $z = -d$, we have

Continuity of $\boldsymbol{E}_\parallel$: $$aE_0\exp(\mathrm{i}n\omega d/c) + bE_0\exp(-\mathrm{i}n\omega d/c) = \tau E_0\exp(\mathrm{i}\omega d/c). \tag{7a}$$

Continuity of $\boldsymbol{H}_\parallel$:

$$Z_0^{-1}[-anE_0\exp(\mathrm{i}n\omega d/c) + bnE_0\exp(-\mathrm{i}n\omega d/c)] = -Z_0^{-1}\tau E_0\exp(\mathrm{i}\omega d/c). \tag{7b}$$

c) Equations (6) and (7) may now be solved to determine the coefficients a, b, ρ, τ, as follows:

$$1+\rho = a+b, \tag{8a}$$

$$1-\rho = (a-b)n, \tag{8b}$$

$$a\exp(\mathrm{i}n\omega d/c) + b\exp(-\mathrm{i}n\omega d/c) = \tau\exp(\mathrm{i}\omega d/c), \tag{8c}$$

$$an\exp(\mathrm{i}n\omega d/c) - bn\exp(-\mathrm{i}n\omega d/c) = \tau\exp(\mathrm{i}\omega d/c). \tag{8d}$$

From the above equations we find

$$\frac{1-\rho}{1+\rho} = n\left[\frac{1-(b/a)}{1+(b/a)}\right], \tag{9a}$$

$$\frac{b}{a} = \left(\frac{n-1}{n+1}\right)\exp(\mathrm{i}2n\omega d/c). \tag{9b}$$

Substituting from Eq.(9b) into Eq.(9a), then solving for ρ, we arrive at

$$\frac{1-\rho}{1+\rho} = n\left[\frac{1-[(n-1)/(n+1)]\exp(\mathrm{i}2n\omega d/c)}{1+[(n-1)/(n+1)]\exp(\mathrm{i}2n\omega d/c)}\right] \quad \rightarrow \quad \rho = \frac{[(n-1)/(n+1)][1-\exp(\mathrm{i}2n\omega d/c)]}{[(n-1)/(n+1)]^2\exp(\mathrm{i}2n\omega d/c)-1}. \qquad (10)$$

Subsequently, the Fresnel transmission coefficient τ is obtained from Eq.(8c) with the aid of Eqs.(8a) and (8b), as follows:

$$\tau\exp(\mathrm{i}\omega d/c) = (a+b)\cos(n\omega d/c) + \mathrm{i}(a-b)\sin(n\omega d/c)$$

$$\rightarrow \quad \tau = [(1+\rho)\cos(n\omega d/c) + \mathrm{i}n^{-1}(1-\rho)\sin(n\omega d/c)]\exp(-\mathrm{i}\omega d/c). \qquad (11)$$

In the above expressions for ρ and τ, one may replace $(n\omega d/c)$ with $(2\pi nd/\lambda_0)$, where λ_0 is the incident beam's vacuum wavelength.

d) With reference to Eq.(10), the reflectance will be zero when $\exp(\mathrm{i}2n\omega d/c) = 1$. This happens when $n\omega d/c$ becomes an integer-multiple of π, or, equivalently, when $2nd/\lambda_0$ becomes an integer. Thus, when the thickness of the slab is an integer-multiple of half-wavelength within the dielectric, i.e., $\lambda_0/2n$, the reflectance of the slab precisely equals zero.

e) From symmetry of Eq.(10), the maximum reflectance must occur halfway between adjacent minima. Thus when the thickness d is an odd-multiple of a quarter-wavelength within the dielectric, i.e., $\lambda_0/4n$, we will have $\exp(\mathrm{i}2n\omega d/c) = -1$, at which point the reflectance will be a maximum, that is,

$$R_{\max} = |\rho|^2 = 4\left(\frac{n-1}{n+1}\right)^2 \Big/ \left[1+\left(\frac{n-1}{n+1}\right)^2\right]^2. \qquad (12)$$

Note: To see the symmetry of Eq.(10), start the phase-angle $(2n\omega d/c)$ at an integer-multiple of 2π, and change it by $\pm\varphi$. You will find that the corresponding values of ρ are conjugates of each other, and that, therefore, the corresponding values of $R = |\rho|^2$ are identical.
