Fall 2014 Written Comprehensive Exam
Opti 501

Solution to Problem 1: The displacement field is defined as $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$, while the magnetic induction is defined as $\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}$. Maxwell's macroscopic equations are written

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\text{free}},$$
$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \frac{\partial \boldsymbol{D}}{\partial t},$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0.$$

b) In terms of the bound electric charge-density $\rho_{\text{bound}}^{(\text{e})} = -\boldsymbol{\nabla} \cdot \boldsymbol{P}$ and bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(\text{e})} = \partial \boldsymbol{P}/\partial t + \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{M}$, the above Maxwell's equations may be rewritten as

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_{\text{free}} + \rho_{\text{bound}}^{(\text{e})},$$
$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})} \right) + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t},$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0.$$

c) Dot-multiplying the second of the above equations into $\boldsymbol{E}$ and the third equation into $\boldsymbol{B}$, then subtracting one from the other, we will find

$$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) = \mu_0 \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})} \right) + \mu_0 \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t},$$
$$\boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t},$$

Subtraction: $$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) - \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = \mu_0 \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})} \right) + \mu_0 \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$

$$\rightarrow \quad -\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \mu_0 \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})} \right) + \tfrac{1}{2} \mu_0 \varepsilon_0 \frac{\partial (\boldsymbol{E} \cdot \boldsymbol{E})}{\partial t} + \tfrac{1}{2} \frac{\partial (\boldsymbol{B} \cdot \boldsymbol{B})}{\partial t}$$

$$\rightarrow \quad \boldsymbol{\nabla} \cdot (\mu_0^{-1} \boldsymbol{E} \times \boldsymbol{B}) + \frac{\partial}{\partial t} (\tfrac{1}{2} \varepsilon_0 \boldsymbol{E} \cdot \boldsymbol{E} + \tfrac{1}{2} \mu_0^{-1} \boldsymbol{B} \cdot \boldsymbol{B}) + \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})} \right) = 0.$$

d) In the above version of the Poynting theorem, the Poynting vector is $\boldsymbol{S} = \mu_0^{-1} \boldsymbol{E} \times \boldsymbol{B}$, the stored energy in the E-field has density $\tfrac{1}{2} \varepsilon_0 \boldsymbol{E} \cdot \boldsymbol{E}$, the stored energy in the B-field has density $\tfrac{1}{2} \mu_0^{-1} \boldsymbol{B} \cdot \boldsymbol{B}$, and the rate of exchange of electromagnetic energy between the fields and the material media is given by $\boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})} \right) = \boldsymbol{E} \cdot (\boldsymbol{J}_{\text{free}} + \partial \boldsymbol{P}/\partial t + \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{M})$.

e) In terms of bound electric charge-density $\rho_{\text{bound}}^{(\text{e})} = -\boldsymbol{\nabla} \cdot \boldsymbol{P}$, bound electric current-density $\boldsymbol{J}_{\text{bound}}^{(\text{e})} = \partial \boldsymbol{P}/\partial t$, bound magnetic charge-density $\rho_{\text{bound}}^{(\text{m})} = -\boldsymbol{\nabla} \cdot \boldsymbol{M}$, and bound magnetic current-density $\boldsymbol{J}_{\text{bound}}^{(\text{m})} = \partial \boldsymbol{M}/\partial t$, Maxwell's equations may be rewritten as

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_{\text{free}} + \rho_{\text{bound}}^{(\text{e})},$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t},$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{J}_{\text{bound}}^{(\text{m})} - \mu_0 \frac{\partial \boldsymbol{H}}{\partial t},$$

$$\mu_0 \boldsymbol{\nabla} \cdot \boldsymbol{H} = \rho_{\text{bound}}^{(\text{m})}.$$

Dot-multiplying the second of the above equations into $\boldsymbol{E}$ and the third equation into $\boldsymbol{H}$, then subtracting one from the other, we find

$$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) = \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t},$$

$$\boldsymbol{H} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} - \mu_0 \boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t},$$

$$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) - \boldsymbol{H} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} + \mu_0 \boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t}$$

$$\rightarrow \qquad -\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} + \tfrac{1}{2}\varepsilon_0 \frac{\partial (\boldsymbol{E} \cdot \boldsymbol{E})}{\partial t} + \tfrac{1}{2}\mu_0 \frac{\partial (\boldsymbol{H} \cdot \boldsymbol{H})}{\partial t}$$

$$\rightarrow \qquad \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) + \frac{\partial}{\partial t}(\tfrac{1}{2}\varepsilon_0 \boldsymbol{E} \cdot \boldsymbol{E} + \tfrac{1}{2}\mu_0 \boldsymbol{H} \cdot \boldsymbol{H}) + \boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) + \boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} = 0.$$

In the above version of the Poynting theorem, the Poynting vector is $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, the stored energy in the E-field has density $\tfrac{1}{2}\varepsilon_0 \boldsymbol{E} \cdot \boldsymbol{E}$, the stored energy in the H-field has density $\tfrac{1}{2}\mu_0 \boldsymbol{H} \cdot \boldsymbol{H}$, the rate of exchange of electromagnetic energy between the E-field and the material media is $\boldsymbol{E} \cdot \left(\boldsymbol{J}_{\text{free}} + \boldsymbol{J}_{\text{bound}}^{(\text{e})}\right) = \boldsymbol{E} \cdot (\boldsymbol{J}_{\text{free}} + \partial \boldsymbol{P}/\partial t)$, and the rate of exchange of electromagnetic energy between the H-field and material media is $\boldsymbol{H} \cdot \boldsymbol{J}_{\text{bound}}^{(\text{m})} = \boldsymbol{H} \cdot \partial \boldsymbol{M}/\partial t$.

Solution to Problem 2: In the following analysis, the polarization of the incident, reflected, and transmitted beams is taken to be along the x-axis, the speed of light in vacuum is denoted by c, and the impedance of free space is Z_0. The numerical value of Z_0 is $\sim 377\Omega$.

a)
$$\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t) = E_0^{(\mathrm{i})}\hat{\boldsymbol{x}} \exp\left[\mathrm{i}\left(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r} - \omega t\right)\right],$$
$$\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t) = H_0^{(\mathrm{i})}\hat{\boldsymbol{y}} \exp\left[\mathrm{i}\left(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r} - \omega t\right)\right],$$
$$\boldsymbol{k}^{(\mathrm{i})} = -(\omega/c)\hat{\boldsymbol{z}}; \qquad H_0^{(\mathrm{i})} = -E_0^{(\mathrm{i})}/Z_0.$$

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},t) = E_0^{(\mathrm{r})}\hat{\boldsymbol{x}} \exp\left[\mathrm{i}\left(\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{r} - \omega t\right)\right],$$
$$\boldsymbol{H}^{(\mathrm{r})}(\boldsymbol{r},t) = H_0^{(\mathrm{r})}\hat{\boldsymbol{y}} \exp\left[\mathrm{i}\left(\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{r} - \omega t\right)\right],$$
$$\boldsymbol{k}^{(\mathrm{r})} = +(\omega/c)\hat{\boldsymbol{z}}; \qquad H_0^{(\mathrm{r})} = +E_0^{(\mathrm{r})}/Z_0.$$

$$\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t) = E_0^{(\mathrm{t})}\hat{\boldsymbol{x}} \exp\left[\mathrm{i}\left(\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{r} - \omega t\right)\right],$$
$$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = H_0^{(\mathrm{t})}\hat{\boldsymbol{y}} \exp\left[\mathrm{i}\left(\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{r} - \omega t\right)\right],$$
$$\boldsymbol{k}^{(\mathrm{t})} = -(n\omega/c)\hat{\boldsymbol{z}}; \qquad H_0^{(\mathrm{t})} = -nE_0^{(\mathrm{t})}/Z_0.$$

b) At normal incidence, the Fresnel reflection and transmission coefficients from vacuum (where $n_0 = 1$) to water (where $n_1 = 1.33$) are given by

$$\rho = \frac{n_0 - n_1}{n_0 + n_1} = -0.14163; \qquad \tau = \frac{2n_0}{n_0 + n_1} = 0.85837.$$

Consequently, $E_0^{(\mathrm{r})} = -0.14163 E_0^{(\mathrm{i})}$, and $E_0^{(\mathrm{t})} = 0.85837 E_0^{(\mathrm{i})}$.

c) The energy flux per unit area per unit time is the time-averaged Poynting vector, that is,

$$\langle \boldsymbol{S} \rangle = ½\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = ½\mathrm{Re}(E_0\hat{\boldsymbol{x}}\times H_0^*\hat{\boldsymbol{y}}) = \pm ½\, n|E_0|^2\hat{\boldsymbol{z}}/Z_0.$$

Thus for the incident beam

$$S_z^{(\mathrm{i})} = -½\left|E_0^{(\mathrm{i})}\right|^2/Z_0,$$

for the reflected beam

$$S_z^{(\mathrm{r})} = +½\left|E_0^{(\mathrm{r})}\right|^2/Z_0 = ½\,(-0.14163)^2\left|E_0^{(\mathrm{i})}\right|^2/Z_0 = ½\,(0.02006)\left|E_0^{(\mathrm{i})}\right|^2/Z_0,$$

and for the transmitted beam

$$S_z^{(\mathrm{t})} = -½\,(1.33)(0.85837)^2\left|E_0^{(\mathrm{i})}\right|^2/Z_0 = -½(0.97994)\left|E_0^{(\mathrm{i})}\right|^2/Z_0.$$

d) Since $0.97994 + 0.02006 = 1.0$, we conclude that the flux of incident energy is equal to the sum of the reflected and transmitted fluxes of energy. Therefore, energy is being conserved.