Fall 2013 Written Comprehensive Exam (Opti 501)

Solution to Problem 1)

a) At the interface of the perfect conductor with the dielectric layer, the tangential E-field must vanish. Therefore, $E_x(x,y,z=0,t)=E_1\sin(\varphi_1)\cos(\omega_0 t)=0$, which leads to $\varphi_1=0$ or π. In what follows, we shall set $\varphi_1=0$.

b) $$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t}\quad\rightarrow\quad\left(\frac{\partial E_x}{\partial z}\right)\hat{\boldsymbol{y}}=-\mu_o\mu(\omega_0)\frac{\partial\boldsymbol{H}}{\partial t}$$

$$\rightarrow E_1k_1\cos(k_1z)\cos(\omega_0t)=-\mu_o\frac{\partial H_y}{\partial t}\quad\rightarrow\quad\boldsymbol{H}(\boldsymbol{r},t)=-\left(\frac{E_1k_1}{\mu_o\omega_0}\right)\hat{\boldsymbol{y}}\cos(k_1z)\sin(\omega_0t).$$

c) $$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\frac{\partial\boldsymbol{D}(\boldsymbol{r},t)}{\partial t}\quad\rightarrow\quad-\left(\frac{\partial H_y}{\partial z}\right)\hat{\boldsymbol{x}}=\varepsilon_o\varepsilon(\omega_0)\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}$$

$$\rightarrow\quad-\left(\frac{E_1k_1^2}{\mu_o\omega_0}\right)\sin(k_1z)\sin(\omega_0t)=-\varepsilon_o\varepsilon(\omega_0)\omega_0E_1\sin(k_1z)\sin(\omega_0t)$$

$$\rightarrow\quad k_1^2=\mu_o\varepsilon_o\varepsilon(\omega_0)\omega_0^2\quad\rightarrow\quad k_1=(\omega_0/c)n(\omega_0).$$

d) Considering that $\rho_{\text{free}}=0$ and $\boldsymbol{D}=\varepsilon_o\varepsilon\boldsymbol{E}$, the first Maxwell equation becomes $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$. We thus have $\boldsymbol{\nabla}\cdot\boldsymbol{E}=\partial E_x/\partial x=0$. Similarly, since $\boldsymbol{B}=\mu_o\mu\boldsymbol{H}$, Maxwell's fourth equation becomes $\boldsymbol{\nabla}\cdot\boldsymbol{H}=0$, which is automatically satisfied given that $\boldsymbol{\nabla}\cdot\boldsymbol{H}=\partial H_y/\partial y=0$.

e) $$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t}\quad\rightarrow\quad\left(\frac{\partial E_x}{\partial z}\right)\hat{\boldsymbol{y}}=-\mu_o\frac{\partial\boldsymbol{H}}{\partial t}$$

$$\rightarrow E_0k_0\cos(k_0z+\varphi_0)\cos(\omega_0t)=-\mu_o\frac{\partial H_y}{\partial t}$$

$$\rightarrow\quad\boldsymbol{H}(\boldsymbol{r},t)=-\left(\frac{E_0k_0}{\mu_o\omega_0}\right)\hat{\boldsymbol{y}}\cos(k_0z+\varphi_0)\sin(\omega_0t).$$

Substitution for $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$ in Maxwell's second equation now yields

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\frac{\partial\boldsymbol{D}(\boldsymbol{r},t)}{\partial t}\quad\rightarrow\quad-\left(\frac{\partial H_y}{\partial z}\right)\hat{\boldsymbol{x}}=\varepsilon_o\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}$$

$$\rightarrow\quad-\left(\frac{E_0k_0^2}{\mu_o\omega_0}\right)\sin(k_0z+\varphi_0)\sin(\omega_0t)=-\varepsilon_o\omega_0E_0\sin(k_0z+\varphi_0)\sin(\omega_0t)$$

$$\rightarrow\quad k_0^2=\mu_o\varepsilon_o\omega_0^2\quad\rightarrow\quad k_0=\omega_0/c.$$

f) At the interface located at $z=d$, both E_x and H_y must be continuous. We thus have

$$\begin{cases}E_1\sin(k_1d)\cos(\omega_0t)=E_0\sin(k_0d+\varphi_0)\cos(\omega_0t)\\-\left(\frac{E_1k_1}{\mu_o\omega_0}\right)\cos(k_1d)\sin(\omega_0t)=-\left(\frac{E_0k_0}{\mu_o\omega_0}\right)\cos(k_0d+\varphi_0)\sin(\omega_0t)\end{cases}$$

$$\rightarrow\quad\begin{cases}E_1\sin(k_1d)=E_0\sin(k_0d+\varphi_0)\\E_1k_1\cos(k_1d)=E_0k_0\cos(k_0d+\varphi_0)\end{cases}\quad\rightarrow\quad\begin{cases}\tan(k_1d)=n(\omega_0)\tan(k_0d+\varphi_0)\\E_1/E_0=\sin(k_0d+\varphi_0)\,/\sin(k_1d).\end{cases}$$

The above equations may now be solved for the values of φ_0 and E_1/E_0.

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Solution to Problem 2)

a) $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\frac{\partial\boldsymbol{D}(\boldsymbol{r},t)}{\partial t} \;\rightarrow\; -\left(\frac{\partial H_y}{\partial z}\right)\hat{\boldsymbol{x}}+\left(\frac{\partial H_y}{\partial x}\right)\hat{\boldsymbol{z}}=\varepsilon_{\text{o}}\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}$

$$\rightarrow\; -H_0k_z\cos(k_xx)\cos(k_zz-\omega_0t)\hat{\boldsymbol{x}}-H_0k_x\sin(k_xx)\sin(k_zz-\omega_0t)\hat{\boldsymbol{z}}=\varepsilon_{\text{o}}\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}$$

$$\rightarrow\; \boldsymbol{E}(\boldsymbol{r},t)=\frac{H_0k_z}{\varepsilon_0\omega_0}\cos(k_xx)\sin(k_zz-\omega_0t)\,\hat{\boldsymbol{x}}-\frac{H_0k_x}{\varepsilon_0\omega_0}\sin(k_xx)\cos(k_zz-\omega_0t)\,\hat{\boldsymbol{z}}.$$

b) $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \;\rightarrow\; \left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\hat{\boldsymbol{y}}=-\mu_{\text{o}}\frac{\partial\boldsymbol{H}}{\partial t}$

$$\rightarrow\left(\frac{H_0k_z^2}{\varepsilon_0\omega_0}\right)\cos(k_xx)\cos(k_zz-\omega_0t)+\left(\frac{H_0k_x^2}{\varepsilon_0\omega_0}\right)\cos(k_xx)\cos(k_zz-\omega_0t)$$

$$=\mu_{\text{o}}H_0\,\omega_0\cos(k_xx)\cos(k_zz-\omega_0t)$$

$$\rightarrow\; k_x^2+k_z^2=\mu_{\text{o}}\varepsilon_{\text{o}}\omega_0^2 \;\rightarrow\; k_x^2+k_z^2=(\omega_0/c)^2.$$

To ensure that both k_x and k_z are real-valued, we may define an angle θ (see the figure) such that $k_x=(\omega_0/c)\sin\theta$ and $k_z=(\omega_0/c)\cos\theta$.

c) The tangential component of the E-field must vanish at the inner surfaces of the perfect conductors. Therefore, the necessary and sufficient condition for the admissibility of the guided mode is $E_z(x=\pm\tfrac{1}{2}d,y,z,t)=0$, which is equivalent to $\sin(\pm\tfrac{1}{2}k_xd)=0$. We must thus have $\tfrac{1}{2}k_xd=m\pi$, where m is an arbitrary integer. In other words, $k_x=(\omega_0/c)\sin\theta=2\pi m/d$, or, equivalently, $\sin\theta=m\lambda_0/d$.

d) The surface-charge-density is equal to $D_\perp$ in the immediate vicinity of the surface, that is, $|\sigma_s|=D_x=\varepsilon_{\text{o}}E_x$. The sign of σ_s is positive if $D_\perp$ exits from the surface, and negative if $D_\perp$ enters into the surface. We have

$$\sigma_s(x=\pm\tfrac{1}{2}d,y,z,t)=\mp(H_0k_z/\omega_0)\cos(\tfrac{1}{2}k_xd)\sin(k_zz-\omega_0t)$$
$$=\mp(-1)^m(H_0k_z/\omega_0)\sin(k_zz-\omega_0t).$$

The surface-current-density is equal in magnitude and perpendicular in direction to $\boldsymbol{H}_\parallel$ in the immediate vicinity of the surface, that is, $|\boldsymbol{J}_s|=H_y$. The direction of the current is related to the direction of the magnetic field via the right-hand rule. We have

$$\boldsymbol{J}_s(x=\pm\tfrac{1}{2}d,y,z,t)=\mp H_0\hat{\boldsymbol{z}}\cos(\tfrac{1}{2}k_xd)\sin(k_zz-\omega_0t)=\mp(-1)^mH_0\hat{\boldsymbol{z}}\sin(k_zz-\omega_0t).$$

e) The charge-current continuity equation at the inner surfaces of the conductors may now be written as follows:

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_s+\frac{\partial\sigma_s}{\partial t}=\frac{\partial J_{sz}}{\partial z}+\frac{\partial\sigma_s}{\partial t}$$
$$=\mp(-1)^m\,H_0\,k_z\cos(k_zz-\omega_0t)\pm(-1)^mH_0k_z\cos(k_zz-\omega_0t)=0.$$
