Solution to Problem 1)

a) Lorenz Gauge: $\boldsymbol{\nabla}\cdot\boldsymbol{A}+\dfrac{1}{c^2}\dfrac{\partial\psi}{\partial t}=0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}-(\mathrm{i}\omega/c^2)\psi_\mathrm{o}=0 \;\rightarrow\; \boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}=(\omega/c^2)\psi_\mathrm{o}.$

b) $\boldsymbol{E}=-\boldsymbol{\nabla}\psi-\dfrac{\partial\boldsymbol{A}}{\partial t} \;\rightarrow\; \boldsymbol{E}(\boldsymbol{r},t)=(-\mathrm{i}\boldsymbol{k}\psi_\mathrm{o}+\mathrm{i}\omega\boldsymbol{A}_\mathrm{o})\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$

$$\rightarrow\; \boldsymbol{E}_\mathrm{o}=\mathrm{i}(\omega\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}\psi_\mathrm{o}) \;\rightarrow\; \boldsymbol{E}_\mathrm{o}=\mathrm{i}\,\omega[\boldsymbol{A}_\mathrm{o}-(c/\omega)^2(\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o})\boldsymbol{k}].$$

$$\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A} \;\rightarrow\; \boldsymbol{B}(\boldsymbol{r},t)=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \;\rightarrow\; \boldsymbol{B}_\mathrm{o}=\mu_\mathrm{o}\boldsymbol{H}_\mathrm{o}=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o}.$$

c) In free space, $\rho_\text{free}=0$, $\boldsymbol{J}_\text{free}=0$, $\boldsymbol{P}=0$, and $\boldsymbol{M}=0$. Consequently, $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}$ and $\boldsymbol{B}=\mu_0\boldsymbol{H}$. Maxwell's equations thus become

i) $\boldsymbol{\nabla}\cdot\varepsilon_\mathrm{o}\boldsymbol{E}=0 \;\rightarrow\; \mathrm{i}\varepsilon_\mathrm{o}\boldsymbol{k}\cdot\boldsymbol{E}_\mathrm{o}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0 \;\rightarrow\; \boldsymbol{k}\cdot\boldsymbol{E}_\mathrm{o}=0 \;\rightarrow\; \boldsymbol{k}\cdot(\omega\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}\psi_\mathrm{o})=0$

$$\rightarrow\; \omega\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}^2\psi_\mathrm{o}=0 \;\rightarrow\; (\omega/c)^2\psi_\mathrm{o}-\boldsymbol{k}^2\psi_\mathrm{o}=0 \;\rightarrow\; \text{Either } \psi_o=0 \text{ or } \boldsymbol{k}^2=(\omega/c)^2.$$

ii) $\boldsymbol{\nabla}\times\boldsymbol{H}=\dfrac{\partial\varepsilon_\mathrm{o}\boldsymbol{E}}{\partial t} \;\rightarrow\; \boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{\nabla}\times\mu_\mathrm{o}\boldsymbol{H}=\dfrac{\partial\boldsymbol{E}}{c^2\partial t} \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o})=-(\mathrm{i}\omega/c^2)\boldsymbol{E}_\mathrm{o}$

$$\rightarrow\; (\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o})\boldsymbol{k}-\boldsymbol{k}^2\boldsymbol{A}_\mathrm{o}=-(\omega/c^2)(\omega\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}\psi_\mathrm{o})$$

$$\rightarrow\; [\boldsymbol{k}^2-(\omega/c)^2]\boldsymbol{A}_\mathrm{o}=[\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}-(\omega/c^2)\psi_\mathrm{o}]\boldsymbol{k}=0 \;\rightarrow\; \boldsymbol{k}^2=(\omega/c)^2.$$

iii) $\boldsymbol{\nabla}\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t} \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times[\mathrm{i}(\omega\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}\psi_\mathrm{o})]=\mathrm{i}\omega(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o})$

$$\rightarrow\; \omega\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o}-(\boldsymbol{k}\times\boldsymbol{k})\psi_\mathrm{o}=\omega\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o} \;\rightarrow\; (\boldsymbol{k}\times\boldsymbol{k})\psi_\mathrm{o}=0 \;\rightarrow\; 0=0.$$

iv) $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\cdot(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o})=0 \;\rightarrow\; (\boldsymbol{k}\times\boldsymbol{k})\cdot\boldsymbol{A}_\mathrm{o}=0 \;\rightarrow\; 0=0.$

Thus, the only constraint on the parameters, aside from the Lorenz gauge $\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}=(\omega/c^2)\psi_\mathrm{o}$, is $\boldsymbol{k}^2=(\omega/c)^2$.

d) The plane-wave is inhomogeneous (or evanescent) when the imaginary component of $\boldsymbol{k}$ is non-zero. The constraint $\boldsymbol{k}^2=(\omega/c)^2$ thus yields

$$\boldsymbol{k}^2=(\omega/c)^2 \;\rightarrow\; (\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')=(\omega/c)^2 \;\rightarrow\; \boldsymbol{k}'^2-\boldsymbol{k}''^2+2\mathrm{i}\,\boldsymbol{k}'\cdot\boldsymbol{k}''=(\omega/c)^2$$

$$\rightarrow\; \boldsymbol{k}'^2-\boldsymbol{k}''^2=(\omega/c)^2 \quad\text{and}\quad \boldsymbol{k}'\cdot\boldsymbol{k}''=0.$$

For the plane-wave to be evanescent, it is thus necessary for $\boldsymbol{k}'$ and $\boldsymbol{k}''$ to be orthogonal to each other. It is also necessary for $|\boldsymbol{k}'|$ to be greater than ω/c, so that $|\boldsymbol{k}''|$ will be real-valued.

e) When $\boldsymbol{k}$ is a real-valued vector, i.e., when $\boldsymbol{k}''=0$, the plane-wave will be homogeneous. Both $\boldsymbol{E}_\mathrm{o}$ and $\boldsymbol{B}_\mathrm{o}$ will then be proportional to the transverse component $\boldsymbol{A}_{\mathrm{o}\perp}=\boldsymbol{A}_\mathrm{o}-(c/\omega)^2(\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o})\boldsymbol{k}$ of $\boldsymbol{A}_\mathrm{o}$, with $\boldsymbol{B}_\mathrm{o}$ rotated around the k-vector by 90º. The plane-wave will be linearly polarized if the real and imaginary parts of this transverse vector potential, namely, $\boldsymbol{A}'_{o\perp}$ and $\boldsymbol{A}''_{o\perp}$, happen to be parallel to each other, or if one of them (either $\boldsymbol{A}'_{o\perp}$ or $\boldsymbol{A}''_{o\perp}$) vanishes.

f) Again, the plane-wave is homogeneous when $\boldsymbol{k}''=0$. As before, $\boldsymbol{E}_o$ and $\boldsymbol{B}_o$ will be proportional to $\boldsymbol{A}_{o\perp}$, with $\boldsymbol{B}_o$ rotated around $\boldsymbol{k}$ by 90º. The plane-wave will be circularly polarized if $\boldsymbol{A}'_{o\perp}$ and $\boldsymbol{A}''_{o\perp}$ happen to be equal in magnitude and perpendicular to each other.

g) The time-averaged rate of flow of electromagnetic energy is given by the time-averaged Poynting vector, that is,

$$
\begin{aligned}
<\boldsymbol{S}(\boldsymbol{r},t)> &= \tfrac{1}{2}\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \tfrac{1}{2}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}(\boldsymbol{E}_o\times\boldsymbol{H}_o^*)\\
&= \tfrac{1}{2}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}\{(\omega\boldsymbol{A}_o-\boldsymbol{k}\psi_o)\times\mu_o^{-1}\boldsymbol{k}^*\times\boldsymbol{A}_o^*\}\\
&= \frac{\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})}{2\mu_o}\,\mathrm{Re}\{[(\boldsymbol{A}_o\cdot\boldsymbol{A}_o^*)\boldsymbol{k}^*-(\boldsymbol{k}^*\cdot\boldsymbol{A}_o)\boldsymbol{A}_o^*]\omega+[(\boldsymbol{k}\cdot\boldsymbol{k}^*)\boldsymbol{A}_o^*-(\boldsymbol{k}\cdot\boldsymbol{A}_o^*)\boldsymbol{k}^*]\psi_o\}.
\end{aligned}
$$

Solution to Problem 2)

a)
$$\boldsymbol{E}(z,t)=E_o\hat{\boldsymbol{x}}\sin(kz-\omega t)+E_o\hat{\boldsymbol{x}}\sin(kz+\omega t)=2E_o\hat{\boldsymbol{x}}\sin(kz)\cos(\omega t).$$

$$\boldsymbol{H}(z,t)=Z_o^{-1}E_o\hat{\boldsymbol{y}}[\sin(kz-\omega t)-\sin(kz+\omega t)]=-2Z_o^{-1}E_o\hat{\boldsymbol{y}}\cos(kz)\sin(\omega t).$$

b) Denoting the electromagnetic energy-density at point z and instant t by $\mathcal{E}(z,t)$, we will have

$$
\begin{aligned}
\mathcal{E}(z,t) &= \tfrac{1}{2}\varepsilon_o\boldsymbol{E}^2(z,t)+\tfrac{1}{2}\mu_o\boldsymbol{H}^2(z,t)=2\varepsilon_oE_o^2\sin^2(kz)\cos^2(\omega t)+2\mu_oZ_o^{-2}E_o^2\cos^2(kz)\sin^2(\omega t)\\
&= 2\varepsilon_oE_o^2\sin^2(kz)(1-\sin^2\omega t)+2\varepsilon_oE_o^2\cos^2(kz)\sin^2(\omega t)\\
&= 2\varepsilon_oE_o^2[\sin^2(kz)+\cos(2kz)\sin^2(\omega t)].
\end{aligned}
$$

Integration of the above energy-density over z from 0 to L yields $L/2$ for $\sin^2(kz)$ and zero for $\cos(2kz)$. The total energy contained in the cavity thus turns out to be $\varepsilon_oE_o^2AL$, independent of time. The number of photons is now found to be $\mathcal{N}=\varepsilon_oE_o^2AL/(\hbar\omega)$. At any given moment, one-half of these photons may be said to be propagating from left to right, while the remaining half travel from right to left.

c) At $z=0$, the magnetic field at the mirror surface is $\boldsymbol{H}(z=0^+,t)=-2Z_o^{-1}E_o\hat{\boldsymbol{y}}\sin(\omega t)$. Since the fields inside a perfect conductor are always zero, Maxwell's boundary conditions require that $\boldsymbol{J}_s(z=0,t)=2Z_o^{-1}E_o\hat{\boldsymbol{x}}\sin(\omega t)$. The effective magnetic field acting on this surface-current is the *average* of the H-fields immediately in front of and immediately beneath the surface, that is, $\boldsymbol{H}_{\text{eff}}(z=0,t)=-Z_o^{-1}E_o\hat{\boldsymbol{y}}\sin(\omega t)$. The Lorentz force-density exerted by this field on the surface current $\boldsymbol{J}_s$ is, therefore, going to be

$$\boldsymbol{J}_s(z=0,t)\times\mu_o\boldsymbol{H}_{\text{eff}}(z=0,t)=2Z_o^{-1}E_o\sin(\omega t)\hat{\boldsymbol{x}}\times[-\mu_oZ_o^{-1}E_o\sin(\omega t)\hat{\boldsymbol{y}}]=-2\varepsilon_oE_o^2\sin^2(\omega t)\hat{\boldsymbol{z}}.$$

Time-averaging and integration over the surface area of the mirror yields a total force equal to $<\boldsymbol{F}(z=0,t)> = -\varepsilon_o E_o^2 A\hat{\boldsymbol{z}}$. A similar procedure (or an argument from symmetry) yields the force on the mirror located at $z=L$ as $<\boldsymbol{F}(z=L,t)> = \varepsilon_o E_o^2 A\hat{\boldsymbol{z}}$.

d) The work done by radiation pressure in moving the mirror a distance Δz adds to the kinetic energy of the mirror. Since the mirror is stationary at first, the work done in moving it from $z=L$ to $z=L+\Delta z$ must be equal to its final kinetic energy, that is, $\frac{1}{2}MV^2 = F_z\Delta z = \varepsilon_0 E_o^2 A\,\Delta z$.

e) For the forward propagating wave inside the cavity, the Poynting vector is given by

$$\boldsymbol{S}(z,t) = \boldsymbol{E}(z,t)\times\boldsymbol{H}(z,t) = E_o\sin(kz-\omega t)\hat{\boldsymbol{x}}\times Z_o^{-1}E_o\sin(kz-\omega t)\hat{\boldsymbol{y}} = Z_o^{-1}E_o^2\sin^2(kz-\omega t)\hat{\boldsymbol{z}}.$$

The electromagnetic momentum density is given by $\boldsymbol{S}(z,t)/c^2$. Integration over the volume of the cavity thus yields the total momentum content of the forward-propagating wave as $\frac{1}{2}\varepsilon_0 E_o^2 AL/c$. Upon reflection from the mirror located at $z=L$, *twice* this momentum will be transferred to the mirror during the time interval $\frac{1}{2}\Delta t = L/c$, which is the time it takes for the entire forward-propagating wave within the cavity to bounce off the front mirror. By this time, the other plane-wave has turned around and is now propagating in the forward direction, so, during the next $\frac{1}{2}\Delta t$ interval, another transfer of momentum (from the electromagnetic field to the front mirror) takes place. The total momentum acquired by the front mirror between $t=t_0$ and $t=t_0+2L/c$ is, therefore, given by $MV = 2\varepsilon_0 E_o^2 AL/c = \varepsilon_0 E_o^2 A\,\Delta t$. This result is also consistent with Newton's law, $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$, when applied to the front mirror under the influence of the electromagnetic force $<\boldsymbol{F}(z=L,t)> = \varepsilon_o E_o^2 A\hat{\boldsymbol{z}}$. Alternatively, one could obtain the momentum of the mirror by multiplying the total number of photons $\mathcal{N}$, computed in part (b), with *twice* the momentum of each photon, $\hbar\omega/c$, thus arriving at $MV = 2\mathcal{N}\hbar\omega/c = 2\varepsilon_0 E_o^2 AL/c$.

f) Since $L = N\lambda/2 = \pi cN/\omega$, we have $\mathrm{d}L/\mathrm{d}\omega = -\pi cN/\omega^2 = -L/\omega$. Therefore, $\Delta z/L = -\Delta\omega/\omega$.

g) In part (d) we found that $\frac{1}{2}MV^2 = \varepsilon_0 E_o^2 A\,\Delta z$. Writing $\frac{1}{2}MV^2 = \frac{1}{2}(MV)V$ and substituting for the momentum MV the result obtained in part (e), namely, $MV = \varepsilon_0 E_o^2 A\,\Delta t$, we find $\Delta z = \frac{1}{2}V\Delta t$. This should not come as a surprise, however, considering that over the short time interval Δt, the radiation pressure exerts a constant force on the front mirror, resulting in a constant acceleration a. Since $\Delta z = \frac{1}{2}a(\Delta t)^2$ and $V = a\Delta t$, it is obvious that Δz must be equal to $\frac{1}{2}V\Delta t$. None of the results obtained thus far require the restriction of Δt to the specific value $2L/c$; in other words, we expect to find the same results for *any* sufficiently small Δt.

Next, we substitute the above Δz into the expression for $\Delta\omega/\omega$ obtained in part (f). We find

$$\Delta\omega/\omega = -\Delta z/L = -\tfrac{1}{2}V\Delta t/L = -V/c.$$

This is the formula for the Doppler shift upon reflection from a flat mirror moving at the constant velocity $\frac{1}{2}V\hat{\boldsymbol{z}}$, which, in the system under consideration, is the *average* mirror velocity during the time interval $\Delta t = 2L/c$. What is special about $\Delta t = 2L/c$ is that it is the time interval during which each and every photon inside the cavity gets exactly one chance to bounce off the front mirror. Without this restriction, one does *not* arrive at the Doppler relation between $\Delta\omega$ and the (average) mirror velocity. The red-shift (or cooling down) of the photons is a direct consequence of their transfer of energy and momentum to the free-standing mirror.

At much later times, when $t \gg t_o$, the free mirror will have an initial velocity V in the beginning of each cycle, reaching $V+\delta V$ after a time interval $\Delta t = 2L/c$. (L is now the length of the cavity in the beginning of the cycle.) The increase in the kinetic energy of the mirror will then be $F_z \Delta z = MV\delta V$, whereas the change in the mirror's momentum will be $F_z \Delta t = M\delta V$. Consequently, $\Delta z = V\Delta t$ and $\Delta\omega/\omega = -\Delta z/L = -2V/c$, which is the Doppler shift upon reflection from a mirror moving at an average velocity $V + ½\delta V$, provided, of course, that δV is negligible compared to V.
