Problem 1)

a) The plane-waves' *E*- and *H*-fields have the following general form:

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{\text{o}}\exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\text{o}}t)],$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{\text{o}}\exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\text{o}}t)].$$

For the incident (i), reflected (r), and transmitted (t) beams we have

$$\boldsymbol{k}^{(\text{i})}=(n_1\omega_{\text{o}}/c)(\sin\theta\hat{\boldsymbol{y}}-\cos\theta\hat{\boldsymbol{z}}),$$
$$\boldsymbol{k}^{(\text{r})}=(n_1\omega_{\text{o}}/c)(\sin\theta\hat{\boldsymbol{y}}+\cos\theta\hat{\boldsymbol{z}}),$$
$$\boldsymbol{k}^{(\text{t})}=(n_2\omega_{\text{o}}/c)(\sin\theta'\hat{\boldsymbol{y}}-\cos\theta'\hat{\boldsymbol{z}}).$$

In what follows, we shall use Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, to relate the *H*-field to the *E*-field via the unit vector $\hat{\boldsymbol{k}}=\boldsymbol{k}/k$ along the propagation direction, namely,

$$\boldsymbol{H}_{\text{o}}=\boldsymbol{k}\times\boldsymbol{E}_{\text{o}}/(\mu_{\text{o}}\omega_{\text{o}})=(n/Z_{\text{o}})\hat{\boldsymbol{k}}\times\boldsymbol{E}_{\text{o}}.$$

Defining the Fresnel reflection and transmission coefficients for *p*- and *s*-light as ρ_p, ρ_s, τ_p, and τ_s, we write

$$\boldsymbol{E}_{\text{o}}^{(\text{i})}=E_s^{(\text{i})}\hat{\boldsymbol{x}}+E_p^{(\text{i})}(\cos\theta\hat{\boldsymbol{y}}+\sin\theta\hat{\boldsymbol{z}}),$$
$$\boldsymbol{H}_{\text{o}}^{(\text{i})}=(n_1/Z_{\text{o}})(\sin\theta\hat{\boldsymbol{y}}-\cos\theta\hat{\boldsymbol{z}})\times\boldsymbol{E}_{\text{o}}^{(\text{i})}=(n_1/Z_{\text{o}})[E_p^{(\text{i})}\hat{\boldsymbol{x}}-E_s^{(\text{i})}(\cos\theta\hat{\boldsymbol{y}}+\sin\theta\hat{\boldsymbol{z}})].$$

$$\boldsymbol{E}_{\text{o}}^{(\text{r})}=\rho_sE_s^{(\text{i})}\hat{\boldsymbol{x}}+\rho_pE_p^{(\text{i})}(\cos\theta\hat{\boldsymbol{y}}-\sin\theta\hat{\boldsymbol{z}}),$$
$$\boldsymbol{H}_{\text{o}}^{(\text{r})}=(n_1/Z_{\text{o}})(\sin\theta\hat{\boldsymbol{y}}+\cos\theta\hat{\boldsymbol{z}})\times\boldsymbol{E}_{\text{o}}^{(\text{r})}=(n_1/Z_{\text{o}})[-\rho_pE_p^{(\text{i})}\hat{\boldsymbol{x}}+\rho_sE_s^{(\text{i})}(\cos\theta\hat{\boldsymbol{y}}-\sin\theta\hat{\boldsymbol{z}})].$$

$$\boldsymbol{E}_{\text{o}}^{(\text{t})}=\tau_sE_s^{(\text{i})}\hat{\boldsymbol{x}}+\tau_pE_p^{(\text{i})}(\cos\theta'\hat{\boldsymbol{y}}+\sin\theta'\hat{\boldsymbol{z}}),$$
$$\boldsymbol{H}_{\text{o}}^{(\text{t})}=(n_2/Z_{\text{o}})(\sin\theta'\hat{\boldsymbol{y}}-\cos\theta'\hat{\boldsymbol{z}})\times\boldsymbol{E}_{\text{o}}^{(\text{t})}=(n_2/Z_{\text{o}})[\tau_pE_p^{(\text{i})}\hat{\boldsymbol{x}}-\tau_sE_s^{(\text{i})}(\cos\theta'\hat{\boldsymbol{y}}+\sin\theta'\hat{\boldsymbol{z}})].$$

Note that the Snell's law requirement, $k_y^{(\text{i})}=k_y^{(\text{r})}=k_y^{(\text{t})}$, is readily satisfied for the reflected beam by setting $\theta^{(\text{r})}=\theta^{(\text{i})}=\theta$, whereas for the transmitted beam we must have $n_1\sin\theta=n_2\sin\theta'$. Also, within the transmission medium, $(k_y^2+k_z^2)^{(\text{t})}=(n_2\omega_{\text{o}}/c)^2$ results in the following relation: $k_z^{(\text{t})}=-(n_2\omega_{\text{o}}/c)\cos\theta'=-(n_2\omega_{\text{o}}/c)\sqrt{1-\sin^2\theta'}$. As long as $\sin\theta'$ is below unity, the argument of the square root will be non-negative and, therefore, the sign of the square root will be positive (by convention). However, when $n_1\sin\theta>n_2$, the square root becomes imaginary, necessitating a choice between + and – for its sign. In the geometry chosen for this problem, we must choose $\cos\theta'=\text{i}\sqrt{\sin^2\theta'-1}=\text{i}\sqrt{(n_1^2\sin^2\theta/n_2^2)-1}$, to ensure that the evanescent wave inside the transmission medium decays exponentially away from the interface. Note also that τ_p is defined slightly differently here than in Chapter 7; here τ_p is the transmission coefficient for E_p, not E_y.

b) To satisfy the boundary conditions we equate the tangential components of the E- and H-fields across the interface. We will have

Continuity of E_x: $E_s^{(\mathrm{i})} + \rho_s E_s^{(\mathrm{i})} = \tau_s E_s^{(\mathrm{i})} \quad \rightarrow \quad 1 + \rho_s = \tau_s,$

Continuity of E_y: $E_p^{(\mathrm{i})} \cos\theta + \rho_p E_p^{(\mathrm{i})} \cos\theta = \tau_p E_p^{(\mathrm{i})} \cos\theta' \quad \rightarrow \quad (1 + \rho_p)\cos\theta = \tau_p \cos\theta',$

Continuity of H_x: $n_1 E_p^{(\mathrm{i})} - n_1 \rho_p E_p^{(\mathrm{i})} = n_2 \tau_p E_p^{(\mathrm{i})} \quad \rightarrow \quad n_1(1 - \rho_p) = n_2 \tau_p,$

Continuity of H_y: $-n_1 E_s^{(\mathrm{i})} \cos\theta + n_1 \rho_s E_s^{(\mathrm{i})} \cos\theta = -n_2 \tau_s E_s^{(\mathrm{i})} \cos\theta' \rightarrow n_1(1 - \rho_s)\cos\theta = n_2 \tau_s \cos\theta'.$

The 1st and 4th of the above equations then yield

$$\rho_s = \frac{n_1 \cos\theta - n_2 \cos\theta'}{n_1 \cos\theta + n_2 \cos\theta'}, \qquad \tau_s = \frac{2 n_1 \cos\theta}{n_1 \cos\theta + n_2 \cos\theta'}.$$

Similarly, from the 2nd and 3rd equations we find

$$\rho_p = \frac{n_1 \cos\theta' - n_2 \cos\theta}{n_1 \cos\theta' + n_2 \cos\theta}, \qquad \tau_p = \frac{2 n_1 \cos\theta}{n_1 \cos\theta' + n_2 \cos\theta}.$$

c) When $\rho_p = 0$ we have $n_1 \cos\theta' = n_2 \cos\theta$, from which, after some algebraic manipulations, we obtain $\tan\theta = n_2/n_1$ and $\tan\theta' = n_1/n_2$. This incidence angle at which the reflectivity of p-polarized light vanishes is known as Brewster's angle, θ_B. There is no Brewster's angle for s-light.

d) When $\cos\theta'$ becomes imaginary, the magnitudes of both ρ_p and ρ_s become unity, that is, $|\rho_p| = |\rho_s| = 1$. This is because these Fresnel coefficients assume the form $(a - \mathrm{i}b)/(a + \mathrm{i}b)$, which, as the ratio of two complex numbers of equal magnitude, has a magnitude of 1. As mentioned earlier, for $\cos\theta'$ to become imaginary, the incidence angle must exceed a certain critical angle, i.e., $n_1 \sin\theta > n_2$, which happens when $n_1 > n_2$ and $\theta > \theta_{\mathrm{crit}} = \arcsin(n_2/n_1)$. These conditions apply to p-light and s-light alike. Beyond the critical angle θ_{crit}, both p- and s-polarized incident beams get totally reflected at the interface, although the phase of the reflection coefficient ρ_p differs from that of ρ_s at any given incidence angle θ.

Problem 2)

a) Continuity of E_x at the entrance facet: $E_x^{(\text{i})} = E_x^{(1)} + E_x^{(2)}$.

Continuity of H_y at the entrance facet: $H_y^{(\text{i})} = H_y^{(1)} + H_y^{(2)} \;\rightarrow\; Z_\text{o}^{-1} E_x^{(\text{i})} = nZ_\text{o}^{-1} E_x^{(1)} - nZ_\text{o}^{-1} E_x^{(2)}$.

We can now solve the above equations for $E_x^{(1)}$ and $E_x^{(2)}$, as follows:

$$E_x^{(1)} = [(n+1)/2n]E_x^{(\text{i})},$$

$$E_x^{(2)} = [(n-1)/2n]E_x^{(\text{i})}.$$

The H-fields are subsequently found to be

$$H_y^{(1)} = [(n+1)/2Z_\text{o}]E_x^{(\text{i})},$$

$$H_y^{(2)} = -[(n-1)/2Z_\text{o}]E_x^{(\text{i})}.$$

The transmitted beam is obtained by matching the boundary conditions at the exit facet of the slab, and using the fact that the magnitude of the k-vector inside the slab is $nk_\text{o} = 2\pi n/\lambda_\text{o}$. We will have

$$E_x^{(\text{t})} = E_x^{(1)}\exp(\text{i}nk_\text{o}d) + E_x^{(2)}\exp(-\text{i}nk_\text{o}d) = E_x^{(1)}\exp(\text{i}\pi) + E_x^{(2)}\exp(-\text{i}\pi) = -E_x^{(1)} - E_x^{(2)} = -E_x^{(\text{i})},$$

$$H_y^{(\text{t})} = H_y^{(1)}\exp(\text{i}nk_\text{o}d) + H_y^{(2)}\exp(-\text{i}nk_\text{o}d) = -H_y^{(1)} - H_y^{(2)} = -E_x^{(\text{i})}/Z_\text{o}.$$

The transmitted field is thus the same as the incident field, albeit with a 180º phase shift.

b)

$$\begin{aligned}
<S_z(z,t)> &= \tfrac{1}{2}\text{Re}\{E_x(z,t)H_y^*(z,t)\} = \tfrac{1}{2}\text{Re}\{\{E_x^{(1)}\exp[\text{i}(nk_\text{o}z-\omega_\text{o}t)] + E_x^{(2)}\exp[\text{i}(-nk_\text{o}z-\omega_\text{o}t)]\}\\
&\qquad \times\{H_y^{(1)}\exp[-\text{i}(nk_\text{o}z-\omega_\text{o}t)] + H_y^{(2)}\exp[-\text{i}(-nk_\text{o}z-\omega_\text{o}t)]\}\}\\
&= \tfrac{1}{2}\text{Re}\{E_x^{(1)}H_y^{(1)} + E_x^{(2)}H_y^{(2)} + E_x^{(1)}H_y^{(2)}\exp(2\text{i}nk_\text{o}z) + E_x^{(2)}H_y^{(1)}\exp(-2\text{i}nk_\text{o}z)\}\\
&= \{(n+1)^2 - (n-1)^2 - 2(n^2-1)\,\text{Re}[\text{i}\sin(2nk_\text{o}z)]\}\,E_x^{(\text{i})2}/(8nZ_\text{o}) = \tfrac{1}{2}E_x^{(\text{i})2}/Z_\text{o}.
\end{aligned}$$

The final result is obviously the same as the time-averaged rate of energy flow (per unit area per unit time) in the incidence medium as well as that in the transmission medium.

c) The above results will remain essentially the same if $d = m\lambda_\text{o}/(2n)$, where $m \neq 1$ is an arbitrary integer. The only change will be in the phase of the transmitted beam when m happens to be *even*, in which case $E_x^{(\text{t})} = E_x^{(\text{i})}$ and $H_y^{(\text{t})} = E_x^{(\text{i})}/Z_\text{o}$.
