Problem 1) a) The *E*- and *H*-fields of the incident plane-wave are given by

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{po}^{(\mathrm{i})}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r}-\omega t)]; \tag{1a}$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{po}^{(\mathrm{i})}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r}-\omega t)]. \tag{1b}$$

The dispersion relation in free space is $k^2=(\omega/c)^2$. Therefore,

$$\boldsymbol{k}^{(\mathrm{i})}=k_x\hat{\boldsymbol{x}}+k_z\hat{\boldsymbol{z}}=(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}}). \tag{2}$$

The incident *E*-field amplitude, as shown in the figure, is given by

$$\boldsymbol{E}_{po}^{(\mathrm{i})}=E_{po}(\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}}). \tag{3}$$

It may be readily verified that this *E*-field satisfies Maxwell's first equation, namely, $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$, which is equivalent to $\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{E}_{po}^{(\mathrm{i})}=0$. As for the incident *H*-field, Maxwell's third equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, yields

$$\mathrm{i}\boldsymbol{k}^{(\mathrm{i})}\times\boldsymbol{E}_{po}^{(\mathrm{i})}=\mathrm{i}\,\omega\mu_o\boldsymbol{H}_{po}^{(\mathrm{i})}\;\rightarrow\;(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}})\times E_{po}(\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}})=\omega\mu_o\boldsymbol{H}_{po}^{(\mathrm{i})}$$

$$\rightarrow\;\boldsymbol{H}_{po}^{(\mathrm{i})}=(E_{po}/Z_o)\,\hat{\boldsymbol{y}}. \tag{4}$$

b) The *E*- and *H*-fields of the reflected wave are written

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{po}^{(\mathrm{r})}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{r}-\omega t)]; \tag{5a}$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{po}^{(\mathrm{r})}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{r}-\omega t)]. \tag{5b}$$

The reflected *k*-vector is similar to the incident *k*-vector, except for the sign of k_z, that is,

$$\boldsymbol{k}^{(\mathrm{r})}=k_x\hat{\boldsymbol{x}}-k_z\hat{\boldsymbol{z}}=(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}}). \tag{6}$$

The reflected *E*-field amplitude must cancel out the tangential component of the incident *E*-field at the surface of the perfect conductor, as there cannot be any *E*-fields inside the conductor. We thus have

$$\boldsymbol{E}_{po}^{(\mathrm{r})}=-E_{po}(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{z}}). \tag{7}$$

As before, it may be readily verified that the above *E*-field satisfies Maxwell's first equation, namely, $\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{E}_{po}^{(\mathrm{r})}=0$. The reflected *H*-field is, once again, obtained from Maxwell's third equation, as follows:

$$\mathrm{i}\boldsymbol{k}^{(\mathrm{r})}\times\boldsymbol{E}_{po}^{(\mathrm{r})}=\mathrm{i}\,\omega\mu_o\boldsymbol{H}_{po}^{(\mathrm{r})}\;\rightarrow\;(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}})\times E_{po}(-\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}})=\omega\mu_o\boldsymbol{H}_{po}^{(\mathrm{r})}$$

$$\rightarrow\;\boldsymbol{H}_{po}^{(\mathrm{r})}=(E_{po}/Z_o)\,\hat{\boldsymbol{y}}. \tag{8}$$

c) The rate of flow of energy per unit cross-sectional area per unit time is given by the time-averaged Poynting vector, namely,

$$<\boldsymbol{S}^{(\mathrm{i})}(\boldsymbol{r},t)> = \tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}^{(\mathrm{i})}_{p\mathrm{o}} \times \boldsymbol{H}^{(\mathrm{i})*}_{p\mathrm{o}}] = \tfrac{1}{2}\mathrm{Re}[E_{p\mathrm{o}}(\cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}}) \times (E^{*}_{p\mathrm{o}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}]$$

$$= \frac{|E_{p\mathrm{o}}|^2}{2Z_{\mathrm{o}}}(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}}). \qquad (9)$$

$$<\boldsymbol{S}^{(\mathrm{r})}(\boldsymbol{r},t)> = \tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}^{(\mathrm{r})}_{p\mathrm{o}} \times \boldsymbol{H}^{(\mathrm{r})*}_{p\mathrm{o}}] = \tfrac{1}{2}\mathrm{Re}[-E_{p\mathrm{o}}(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}}) \times (E^{*}_{p\mathrm{o}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}]$$

$$= \frac{|E_{p\mathrm{o}}|^2}{2Z_{\mathrm{o}}}(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}}). \qquad (10)$$

The incident and reflected waves are seen to have a time-averaged Poynting vector $<\boldsymbol{S}>$ directed along the corresponding k-vector. The magnitudes of these Poynting vectors, however, are the same, namely, $|E_{p\mathrm{o}}|^2/(2Z_{\mathrm{o}})$. Therefore, the incident and reflected energy fluxes are identical.

d) The surface-current-density $\boldsymbol{J}_s(x,y,t)$ is equal in magnitude and perpendicular in direction to the total H-field at the surface of the perfect conductor. Taking into account the right-hand rule relating the direction of the surface current to that of the H-field, we will have

$$\boldsymbol{J}_s(x,y,t) = [H_y^{(\mathrm{i})}(x,y,z=0,t) + H_y^{(\mathrm{r})}(x,y,z=0,t)]\,\hat{\boldsymbol{x}}$$

$$= 2(E_{p\mathrm{o}}/Z_{\mathrm{o}})\exp[\mathrm{i}(k_x x - \omega t)]\,\hat{\boldsymbol{x}} = 2(E_{p\mathrm{o}}/Z_{\mathrm{o}})\exp[\mathrm{i}(\omega/c)(x\sin\theta - ct)]\,\hat{\boldsymbol{x}}. \qquad (11)$$

e) The surface-charge-density $\sigma_s(x,y,t)$ is given by the discontinuity in the perpendicular component of the D-field, that is,

$$\sigma_s(x,y,t) = -\varepsilon_{\mathrm{o}}[E_z^{(\mathrm{i})}(x,y,z=0,t) + E_z^{(\mathrm{r})}(x,y,z=0,t)]$$

$$= 2\varepsilon_{\mathrm{o}}E_{p\mathrm{o}}\sin\theta\exp[\mathrm{i}(k_x x - \omega t)] = 2\varepsilon_{\mathrm{o}}E_{p\mathrm{o}}\sin\theta\exp[\mathrm{i}(\omega/c)(x\sin\theta - ct)]. \qquad (12)$$

f) Substituting in the continuity equation for $\boldsymbol{J}_s$ from Eq.(11) and for σ_s from Eq.(12), we find

$$\partial J_{sx}/\partial x + \partial\sigma_s/\partial t = [2\mathrm{i}(\omega/c)\sin\theta(E_{p\mathrm{o}}/Z_{\mathrm{o}}) - 2\mathrm{i}\,\omega\varepsilon_{\mathrm{o}}E_{p\mathrm{o}}\sin\theta]\exp[\mathrm{i}(\omega/c)(x\sin\theta - ct)]$$

$$= 2\mathrm{i}\omega[(cZ_{\mathrm{o}})^{-1} - \varepsilon_{\mathrm{o}}]E_{p\mathrm{o}}\sin\theta\exp[\mathrm{i}(\omega/c)(x\sin\theta - ct)] = 0. \qquad (13)$$

The continuity equation is thus satisfied by the induced surface-charge and surface-current.

Problem 2) a) The E- and H-fields of the incident plane-wave are given by

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{\text{so}}^{(\text{i})}\exp[\text{i}(\boldsymbol{k}^{(\text{i})}\cdot\boldsymbol{r}-\omega t)]; \tag{1a}$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{\text{so}}^{(\text{i})}\exp[\text{i}(\boldsymbol{k}^{(\text{i})}\cdot\boldsymbol{r}-\omega t)]. \tag{1b}$$

The dispersion relation in free space is $k^2=(\omega/c)^2$. Therefore,

$$\boldsymbol{k}^{(\text{i})}=k_x\hat{\boldsymbol{x}}+k_z\hat{\boldsymbol{z}}=(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}}). \tag{2}$$

The incident E-field amplitude, as shown in the figure, is given by

$$\boldsymbol{E}_{\text{so}}^{(\text{i})}=E_{\text{so}}^{(\text{i})}\hat{\boldsymbol{y}}. \tag{3}$$

It may be readily verified that this E-field satisfies Maxwell's first equation, namely, $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$, which is equivalent to $\boldsymbol{k}^{(\text{i})}\cdot\boldsymbol{E}_{\text{so}}^{(\text{i})}=0$. As for the incident H-field, Maxwell's third equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, yields

$$\text{i}\boldsymbol{k}^{(\text{i})}\times\boldsymbol{E}_{\text{so}}^{(\text{i})}=\text{i}\,\omega\mu_{\text{o}}\boldsymbol{H}_{\text{so}}^{(\text{i})} \;\rightarrow\; (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}})\times E_{\text{so}}^{(\text{i})}\hat{\boldsymbol{y}}=\omega\mu_{\text{o}}\boldsymbol{H}_{\text{so}}^{(\text{i})}$$

$$\rightarrow\; \boldsymbol{H}_{\text{so}}^{(\text{i})}=-(E_{\text{so}}^{(\text{i})}/Z_{\text{o}})(\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}}). \tag{4}$$

A similar treatment yields for the reflected plane-wave,

$$\boldsymbol{k}^{(\text{r})}=k_x\hat{\boldsymbol{x}}-k_z\hat{\boldsymbol{z}}=(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}}), \tag{5}$$

$$\boldsymbol{E}_{\text{so}}^{(\text{r})}=E_{\text{so}}^{(\text{r})}\hat{\boldsymbol{y}}, \tag{6}$$

$$\boldsymbol{H}_{\text{so}}^{(\text{r})}=(E_{\text{so}}^{(\text{r})}/Z_{\text{o}})(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{z}}). \tag{7}$$

As for the transmitted beam, the dispersion relation in the dielectric medium is $k^2=(\omega/c)^2n^2(\omega)$; also, in accordance with Snell's law, we must have $k_x^{(\text{t})}=k_x^{(\text{i})}$ and $k_y^{(\text{t})}=k_y^{(\text{i})}=0$. Therefore,

$$\boldsymbol{k}^{(\text{t})}=k_x^{(\text{i})}\hat{\boldsymbol{x}}+k_z^{(\text{t})}\hat{\boldsymbol{z}}=(\omega/c)\left[\sin\theta\,\hat{\boldsymbol{x}}+\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{z}}\right]. \tag{8}$$

Next, we obtain the transmitted E-field using the continuity of tangential $\boldsymbol{E}$ at the interface:

$$\boldsymbol{E}_{\text{so}}^{(\text{t})}=(E_{\text{so}}^{(\text{i})}+E_{\text{so}}^{(\text{r})})\hat{\boldsymbol{y}}. \tag{9}$$

Subsequently, the transmitted H-field is obtained from Maxwell's third equation, as follows:

$$\boldsymbol{k}^{(\text{t})}\times\boldsymbol{E}_{\text{so}}^{(\text{t})}=\omega\mu_{\text{o}}\boldsymbol{H}_{\text{so}}^{(\text{t})} \;\rightarrow\; (\omega/c)\left[\sin\theta\,\hat{\boldsymbol{x}}+\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{z}}\right]\times E_{\text{so}}^{(\text{t})}\hat{\boldsymbol{y}}=\omega\mu_{\text{o}}\boldsymbol{H}_{\text{so}}^{(\text{t})}$$

$$\rightarrow\; \boldsymbol{H}_{\text{so}}^{(\text{t})}=-(E_{\text{so}}^{(\text{t})}/Z_{\text{o}})\left[\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}}\right]. \tag{10}$$

b) Continuity of the tangential E-field is already assured by means of Eq. (9). The only remaining constraint involves the tangential H-field, whose continuity equation is written

$$H_x^{(\text{i})}+H_x^{(\text{r})}=H_x^{(\text{t})} \;\rightarrow\; -(E_{\text{so}}^{(\text{i})}/Z_{\text{o}})\cos\theta+(E_{\text{so}}^{(\text{r})}/Z_{\text{o}})\cos\theta=-(E_{\text{so}}^{(\text{t})}/Z_{\text{o}})\sqrt{n^2(\omega)-\sin^2\theta}\,. \tag{11}$$

The Fresnel reflection and transmission coefficients, defined as $\rho_s=E_{\text{so}}^{(\text{r})}/E_{\text{so}}^{(\text{i})}$ and $\tau_s=E_{\text{so}}^{(\text{t})}/E_{\text{so}}^{(\text{i})}$, may now be used in conjunction with Eqs. (9) and (11) to yield

$$-E_{\text{so}}^{(\text{i})}\cos\theta+\rho_s E_{\text{so}}^{(\text{i})}\cos\theta=-(1+\rho_s)E_{\text{so}}^{(\text{i})}\sqrt{n^2(\omega)-\sin^2\theta}\,. \tag{12}$$

Solving the above equation for ρ_s, we find

$$\rho_s=\frac{\cos\theta-\sqrt{n^2(\omega)-\sin^2\theta}}{\cos\theta+\sqrt{n^2(\omega)-\sin^2\theta}}. \tag{13}$$

From Eq.(9) and the definitions of the Fresnel coefficients, it is obvious that $\tau_s=1+\rho_s$; therefore,

$$\tau_s=\frac{2\cos\theta}{\cos\theta+\sqrt{n^2(\omega)-\sin^2\theta}}. \tag{14}$$

c) The rate-of-flow of energy per unit cross-sectional area per unit time for each of the three plane-waves is given by the corresponding time-averaged Poynting vector, as follows:

$$\begin{aligned}<\boldsymbol{S}^{(\text{i})}(\boldsymbol{r},t)>&=\tfrac{1}{2}\text{Re}[\boldsymbol{E}_{\text{so}}^{(\text{i})}\times\boldsymbol{H}_{\text{so}}^{(\text{i})*}]=-\tfrac{1}{2}\text{Re}[E_{\text{so}}^{(\text{i})}\hat{\boldsymbol{y}}\times(E_{\text{so}}^{(\text{i})*}/Z_{\text{o}})(\cos\theta\,\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}})\\&=\frac{|E_{\text{so}}^{(\text{i})}|^2}{2Z_{\text{o}}}(\sin\theta\,\hat{\boldsymbol{x}}+\cos\theta\,\hat{\boldsymbol{z}}).\end{aligned} \tag{15}$$

$$\begin{aligned}<\boldsymbol{S}^{(\text{r})}(\boldsymbol{r},t)>&=\tfrac{1}{2}\text{Re}[\boldsymbol{E}_{\text{so}}^{(\text{r})}\times\boldsymbol{H}_{\text{so}}^{(\text{r})*}]=\tfrac{1}{2}\text{Re}[E_{\text{so}}^{(\text{r})}\hat{\boldsymbol{y}}\times(E_{\text{so}}^{(\text{r})*}/Z_{\text{o}})(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{z}})\\&=\frac{|E_{\text{so}}^{(\text{r})}|^2}{2Z_{\text{o}}}(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}})=|\rho_s|^2\frac{|E_{\text{so}}^{(\text{i})}|^2}{2Z_{\text{o}}}(\sin\theta\,\hat{\boldsymbol{x}}-\cos\theta\,\hat{\boldsymbol{z}}).\end{aligned} \tag{16}$$

$$\begin{aligned}<\boldsymbol{S}^{(\text{t})}(\boldsymbol{r},t)>&=\tfrac{1}{2}\text{Re}[\boldsymbol{E}_{\text{so}}^{(\text{t})}\times\boldsymbol{H}_{\text{so}}^{(\text{t})*}]=-\tfrac{1}{2}\text{Re}[E_{\text{so}}^{(\text{t})}\hat{\boldsymbol{y}}\times(E_{\text{so}}^{(\text{t})*}/Z_{\text{o}})[\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{x}}-\sin\theta\,\hat{\boldsymbol{z}}]\\&=\frac{|E_{\text{so}}^{(\text{t})}|^2}{2Z_{\text{o}}}[\sin\theta\,\hat{\boldsymbol{x}}+\sqrt{n^2(\omega)-\sin^2\theta}\;\hat{\boldsymbol{z}}]\\&=|\tau_s|^2\frac{n(\omega)|E_{\text{so}}^{(\text{i})}|^2}{2Z_{\text{o}}}(\sin\theta'\,\hat{\boldsymbol{x}}+\cos\theta'\,\hat{\boldsymbol{z}}).\end{aligned} \tag{17}$$

To verify the conservation of energy, consider an incident beam whose cross-sectional diameter in the xz-plane is D. The footprint of this beam on the x-axis will then be $D/\cos\theta$, resulting in a transmitted beam whose cross-sectional diameter in the xz-plane is $D(\cos\theta'/\cos\theta)$. Considering the various Poynting vectors in Eqs.(15)-(17), and the fact that the reflected beam diameter in the xz-plane remains equal to D, we must show that the following identity holds:

$$|\rho_s|^2+(\cos\theta'/\cos\theta)\,n(\omega)\,|\tau_s|^2=1. \tag{18}$$

Substitution from Eqs.(13) and (14) into Eq.(18), and noting that $n(\omega)\cos\theta'=\sqrt{n^2(\omega)-\sin^2\theta}$, then yields

$$\frac{[\cos\theta-\sqrt{n^2(\omega)-\sin^2\theta}\,]^2}{[\cos\theta+\sqrt{n^2(\omega)-\sin^2\theta}\,]^2}+\frac{[\sqrt{n^2(\omega)-\sin^2\theta}\,/\cos\theta]\,(2\cos\theta)^2}{[\cos\theta+\sqrt{n^2(\omega)-\sin^2\theta}\,]^2}=1. \tag{19}$$

The energy fluxes of the reflected and transmitted beams thus add up to that of he incident beam, proving that electromagnetic energy in the present problem is conserved.
