1)a) In Maxwell's first equation, $\vec{\nabla}\cdot\vec{D}=\rho$, the density of free charges is denoted by ρ. $\vec{D}$ is the displacement field, $\vec{D}=\epsilon_0\vec{E}+\vec{P}$ where $\vec{P}$ is the polarization density in material media. In free space, $\vec{P}=0$ and, therefore $\vec{D}=\epsilon_0\vec{E}$. In general $\vec{\nabla}\cdot\vec{E}=\rho_{Total}/\epsilon_0$, where $\rho_{Total}=\rho_{free}+\rho_{bound}$. The density of bound charges is given by $\vec{\nabla}\cdot\vec{P}=-\rho_{bound}$. In these equations $\vec{\nabla}\cdot$ is the divergence operator; for a small volume ΔV, $\vec{\nabla}\cdot\vec{D}$ is the integral of $\vec{D}$ over the surface which encloses the small volume, $\oint\vec{D}\cdot\vec{ds}$, divided by ΔV. In a Cartesian coordinate system $\vec{\nabla}\cdot\vec{D}=\frac{\partial}{\partial x}D_x+\frac{\partial}{\partial y}D_y+\frac{\partial}{\partial z}D_z$.

b) In Maxwell's 2nd equation, $\vec{\nabla}\times\vec{H}=\vec{J}+\partial\vec{D}/\partial t$, the density of free current is denoted by $\vec{J}$. If free charge density at a given point in space and time is $\rho(\vec{r},t)$, and if this charge is moving with velocity $\vec{V}(\vec{r},t)$, then the current density at that point is given by $\vec{J}(\vec{r},t)=\rho(\vec{r},t)\,\vec{V}(\vec{r},t)$.

$\vec{H}$ is the magnetic field; in MKSA it has units of Amp/meter. The curl operator, $\vec{\nabla}\times$, is a vector differential operator. To determine $\vec{\nabla}\times\vec{H}$, one must repeat the following operation for 3 different directions in space (the directions must be mutually orthogonal): Take a small surface element $\vec{\Delta s}$, integrate $\vec{H}$ around the boundary of this surface element, i.e., evaluate $\oint\vec{H}\cdot\vec{dl}$; divide the resulting integral by the area of the surface element, Δs; the resulting value is the magnitude of $\vec{\nabla}\times\vec{H}$ along the direction $\vec{\Delta s}$ (i.e., perpendicular to the surface element). When the operation is repeated for $\vec{\Delta s}_1$, $\vec{\Delta s}_2$, $\vec{\Delta s}_3$ which are mutually orthogonal, the three components of $\vec{\nabla}\times\vec{H}$ will have been determined. Maxwell's 2nd equation states that the curl of the $\vec{H}$-field thus determined is equal to the sum of $\vec{J}$, the density of free current, and the time derivative of the displacement vector, $\partial\vec{D}/\partial t$, at each point in space and time. Note that $\vec{J}+\frac{\partial\vec{D}}{\partial t}=\left(\vec{J}+\frac{\partial\vec{P}}{\partial t}\right)+\epsilon_0\frac{\partial\vec{E}}{\partial t}$. Since $\partial\vec{P}/\partial t$ is the density of bound currents, $\vec{J}_{bound}$, one can write $\vec{\nabla}\times\vec{H}=\vec{J}_{total}+\epsilon_0\frac{\partial\vec{E}}{\partial t}$, that is, the free and bound currents can be ~~lumped~~ together to produce the total current density $\vec{J}_{total}$, which is due to the movement of all charges (i.e., free as well as bound charges).

c) Taking the divergence of the 2nd equation, we'll have:

$$\nabla\cdot(\vec{\nabla}\times\vec{H})=\vec{\nabla}\cdot\vec{J}+\vec{\nabla}\cdot\left(\frac{\partial\vec{D}}{\partial t}\right)\Rightarrow 0=\vec{\nabla}\cdot\vec{J}+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}).$$

From the 1st equation, $\vec{\nabla}\cdot\vec{D}=\rho$. Therefore, $\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$, which is the equation of charge continuity. It states that any net flow of currents into a small volume element is not lost, but results in a change of the local charge density.

d) $\vec{B}=\mu_0(\vec{H}+\vec{M})$ is the magnetic induction. In MKSA the units of $\vec{B}$ are Tesla or Weber/m². $\mu_0=4\pi\times10^{-7}$ is the permeability of the free-space. $\vec{M}$ is the magnetization density of the material medium. Maxwell's third equation, also known as Faraday's law, states that the curl of the $\vec{E}$-field at any location in space and time is exactly equal, but opposite in direction to, the time derivative of the local $\vec{B}$-field. In free space, where $\vec{M}=0$, $\vec{B}=\mu_0\vec{H}$.

Whenever $\partial \vec{B}/\partial t = 0$ (i.e., $\vec{B}$ not varying in time) we'll have $\vec{\nabla} \times \vec{E} = 0$. Using Stokes theorem, we conclude that $\oint \vec{E} \cdot d\vec{l} = 0$ around any closed loop, through which the total magnetic flux is unchanging. Now, $\oint \vec{E} \cdot d\vec{l} = 0$ means that one can evaluate the potential diffrence between points A and B as $\int_A^B \vec{E} \cdot d\vec{l}$ without specifying the path (any path from A to B will yield the same result). This is the condition necessary for defining ψ such that $\vec{E} = -\vec{\nabla}\psi$. Therefore, for the scalar potential ψ to exist such that $\vec{E} = -\vec{\nabla}\psi$, it is sufficient to have $\partial \vec{B}/\partial t = 0$ everywhere.

e) Maxwell's 4th equation, $\vec{\nabla} \cdot \vec{B} = 0$, tells us that whatever $\vec{B}$-field flows into a closed surface, the same amount will flow out of that surface, so that there is no net flux of $\vec{B}$ into or out of any closed surface. The lines of the $\vec{B}$-field, therefore, cannot terminate, nor can they originate, at any point in space. This means that there are no sources or sinks for the $\vec{B}$-field, in other words, there are no magnetic monopoles.

f) Taking the divergece of the 3rd equation, we'll have:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \cdot (\partial \vec{B}/\partial t) \Rightarrow 0 = -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B}) \Rightarrow \vec{\nabla} \cdot \vec{B} = \text{Constant in time.}$$

The 4th equation tells us that the value of the above constant is zero. Thus the 3rd and 4th equations are consistent. However, the 4th equation is <u>not</u> implicit in the 3rd, because the 3rd equation allows $\vec{\nabla} \cdot \vec{B}$ to be an arbitrary function of the position $\vec{r}$ in space. It just says that $\vec{\nabla} \cdot \vec{B}$ is time-independent, which is quite different than saying $\vec{\nabla} \cdot \vec{B} = 0$.

2) a) Incident beam: $\vec{H}^{(i)} = H_{0y}^{(i)} \exp\{i\frac{2\pi}{\lambda}(x\sin\theta + z\cos\theta - ct)\}\hat{y}$

Reflected beam: $\vec{H}^{(r)} = H_{0y}^{(r)} \exp\{i\frac{2\pi}{\lambda}(x\sin\theta - z\cos\theta - ct)\}\hat{y}$

Transmitted beam: $\vec{H}^{(t)} = H_{0y}^{(t)} \exp\{i\frac{2\pi}{\lambda}(nx\sin\theta' + nz\cos\theta' - ct)\}\hat{y}$

To find the E-fields we use $\vec{\nabla}\times\vec{H} = \partial\vec{D}/\partial t = -i\frac{2\pi c}{\lambda}\vec{D} = -i\frac{2\pi c}{\lambda}\epsilon_0\epsilon\vec{E}$.

For the incident beam: $\vec{\nabla}\times\vec{H} = \frac{\partial H_y}{\partial x}\hat{z} - \frac{\partial H_y}{\partial z}\hat{x} = i\frac{2\pi}{\lambda}H_{0y}^{(i)}(\sin\theta\hat{z} - \cos\theta\hat{x})\exp\{\cdots\}$

$\Rightarrow \vec{E}^{(i)} = Z_0 H_{0y}^{(i)}(\cos\theta\,\hat{x} - \sin\theta\,\hat{z})\exp\{i\frac{2\pi}{\lambda}(x\sin\theta + z\cos\theta - ct)\}$

Similarly,

$$\vec{E}^{(r)} = -Z_0 H_{0y}^{(r)}(\cos\theta\,\hat{x} + \sin\theta\,\hat{z})\exp\{i\frac{2\pi}{\lambda}(x\sin\theta - z\cos\theta - ct)\}$$

$$\vec{E}^{(t)} = \frac{Z_0}{n}H_{0y}^{(t)}(\cos\theta'\,\hat{x} - \sin\theta'\,\hat{z})\exp\{i\frac{2\pi}{\lambda}(nx\sin\theta' + nz\cos\theta' - ct)\}$$

b) At the surface, where $z=0$, the complex exponentials must all be the same. Therefore, $\sin\theta = n\sin\theta'$, which is simply the statement of Snell's law. We also need the continuity of E_x and H_y at $z=0$, namely,

$$E_x^{(i)} + E_x^{(r)} = E_x^{(t)} \Rightarrow Z_0\cos\theta\, H_{0y}^{(i)} - Z_0\cos\theta\, H_{0y}^{(r)} = \frac{Z_0}{n}\cos\theta'\, H_{0y}^{(t)}$$

$$H_y^{(i)} + H_y^{(r)} = H_y^{(t)} \Rightarrow H_{0y}^{(i)} + H_{0y}^{(r)} = H_{0y}^{(t)}$$

If the continuity of D_z at $z=0$ is written down, one finds the relationship between the various E_z components as well, but this is not necessary, as the continuity of E_x and H_y provides <u>all</u> the needed relations.

c) The Fresnel Coefficients are usually defined for the E-fields, but here we will derive them for the H-fields. It is not difficult to convert these to the E-field reflection and transmission coefficients, however.

Let $H_{0y}^{(r)} = r H_{0y}^{(i)}$ and $H_{0y}^{(t)} = t H_{0y}^{(i)}$. The Continuity equations at the surface may then be written as follows:

$$\begin{cases} \cos\theta\,(1-r) = \frac{t}{n}\cos\theta' \\ 1+r=t \end{cases} \Rightarrow \cos\theta - r\cos\theta = \frac{\cos\theta'}{n} + \frac{r\cos\theta'}{n} \Rightarrow$$

$$r = \frac{n\cos\theta - \cos\theta'}{n\cos\theta + \cos\theta'}, \quad t = 1+r = \frac{2n\cos\theta}{n\cos\theta + \cos\theta'}$$

The Fresnel coefficients for the E-field are similar to r and t above, except for a sign change in r and the elimination of $\underline{n}$ from the numerator of t.