

1) a) In Maxwell's first equation, $\vec{\nabla} \cdot \vec{D} = \rho$, the density of free charges is denoted by ρ . \vec{D} is the displacement field, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ where \vec{P} is the polarization density in material media. In free space, $\vec{P} = 0$ and, therefore $\vec{D} = \epsilon_0 \vec{E}$. In general $\vec{\nabla} \cdot \vec{E} = \rho_{\text{total}} / \epsilon_0$, where $\rho_{\text{total}} = \rho_{\text{free}} + \rho_{\text{bound}}$. The density of bound charges is given by $\vec{\nabla} \cdot \vec{P} = -\rho_{\text{bound}}$. In these equations $\vec{\nabla} \cdot$ is the divergence operator; for a small volume ΔV , $\vec{\nabla} \cdot \vec{D}$ is the integral of \vec{D} over the surface which encloses the small volume, $\oint \vec{D} \cdot d\vec{s}$, divided by ΔV . In a Cartesian coordinate system $\vec{\nabla} \cdot \vec{D} = \frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} D_z$.

b) In Maxwell's 2nd equation, $\vec{\nabla} \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t$, the density of free current is denoted by \vec{J} . If free charge density at a given point in space and time is $\rho(\vec{r}, t)$, and if this charge is moving with velocity $\vec{v}(\vec{r}, t)$, then the current density at that point is given by $\vec{J}(\vec{r}, t) = \rho(\vec{r}, t) \vec{v}(\vec{r}, t)$.

\vec{H} is the magnetic field; in MKSA it has units of Amp/meter. The curl operator, $\vec{\nabla} \times$, is a vector differential operator. To determine $\vec{\nabla} \times \vec{H}$, one must repeat the following operation for 3 different directions in space (the directions must be mutually orthogonal): Take a small surface element $\Delta \vec{s}$, integrate \vec{H} around the boundary of this surface element, i.e., evaluate $\oint \vec{H} \cdot d\vec{\ell}$; divide the resulting integral by the area of the surface element, Δs ; the resulting value is the magnitude of $\vec{\nabla} \times \vec{H}$ along the direction $\Delta \vec{s}$ (i.e., perpendicular to the surface element). When the operation is repeated for $\Delta \vec{s}_1, \Delta \vec{s}_2, \Delta \vec{s}_3$, which are mutually orthogonal, the three components of $\vec{\nabla} \times \vec{H}$ will have

been determined. Maxwell's 2nd equation states that the curl of the \vec{H} -field thus determined is equal to the sum of \vec{J} , the density of free current, and the time derivative of the displacement vector, $\partial \vec{D} / \partial t$, at each point in space and time. Note that $\vec{J} + \frac{\partial \vec{D}}{\partial t} = (\vec{J} + \frac{\partial \vec{P}}{\partial t}) + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. Since $\partial \vec{P} / \partial t$ is the density of bound currents, \vec{J}_{bound} , one can write $\vec{\nabla} \times \vec{H} = \vec{J}_{\text{total}} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$, that is, the free and bound currents can be lumped together to produce the total current density \vec{J}_{total} , which is due to the movement of all charges (i.e., free as well as bound charges).

c) Taking the divergence of the 2nd equation, we'll have:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \left(\frac{\partial \vec{D}}{\partial t} \right) \Rightarrow 0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{D})$$

From the 1st equation, $\vec{\nabla} \cdot \vec{D} = \rho$. Therefore, $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$, which is the equation of charge continuity. It states that any net flow of currents into a small volume element is not lost, but results in a change of the local charge density.

d) $\vec{B} = \mu_0 (\vec{H} + \vec{M})$ is the magnetic induction. In MKSA the units of \vec{B} are Tesla or Weber/m². $\mu_0 = 4\pi \times 10^{-7}$ is the permeability of the free-space. \vec{M} is the magnetization density of the material medium. Maxwell's third equation, also known as Faraday's law, states that the curl of the \vec{E} -field at any location in space and time is exactly equal, but opposite in direction to, the time derivative of the local \vec{B} -field. In free space, where $\vec{M} = 0$, $\vec{B} = \mu_0 \vec{H}$.

Whenever $\partial \vec{B} / \partial t = 0$ (i.e., \vec{B} not varying in time) we'll have $\vec{\nabla} \times \vec{E} = 0$. Using Stokes theorem, we conclude that $\oint \vec{E} \cdot d\vec{\ell} = 0$ around any closed loop, through which the total magnetic flux is unchanging. Now, $\oint \vec{E} \cdot d\vec{\ell} = 0$ means that one can evaluate the potential difference between points A and B as $\int_A^B \vec{E} \cdot d\vec{\ell}$ without specifying the path (any path from A to B will yield the same result). This is the condition necessary for defining ψ such that $\vec{E} = -\vec{\nabla} \psi$. Therefore, for the scalar potential ψ to exist such that $\vec{E} = -\vec{\nabla} \psi$, it is sufficient to have $\partial \vec{B} / \partial t = 0$ everywhere.

e) Maxwell's 4th equation, $\vec{\nabla} \cdot \vec{B} = 0$, tells us that whatever \vec{B} -field flows into a closed surface, the same amount will flow out of that surface, so that there is no net flux of \vec{B} into or out of any closed surface. The lines of the \vec{B} -field, therefore, cannot terminate, nor can they originate, at any point in space. This means that there are no sources or sinks for the \vec{B} -field, in other words, there are no magnetic monopoles.

f) Taking the divergence of the 3rd equation, we'll have:

$$\nabla \cdot (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \cdot (\partial \vec{B} / \partial t) \Rightarrow 0 = -\frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{B}) \Rightarrow \vec{\nabla} \cdot \vec{B} = \text{constant in time.}$$

The 4th equation tells us that the value of the above constant is zero. Thus the 3rd and 4th equations are consistent. However, the 4th equation is not implicit in the 3rd, because the 3rd equation allows $\vec{\nabla} \cdot \vec{B}$ to be an arbitrary function of the position \vec{r} in space. It just says that $\vec{\nabla} \cdot \vec{B}$ is time-independent, which is quite different than saying $\vec{\nabla} \cdot \vec{B} = 0$.

c) The Fresnel coefficients are usually defined for the E-fields, but here we will derive them for the H-fields. It is not difficult to convert these to the E-field reflection and transmission coefficients, however.

Let $H_{oy}^{(r)} = r H_{oy}^{(i)}$ and $H_{oy}^{(t)} = t H_{oy}^{(i)}$. The continuity equations at the surface may then be written as follows:

$$\begin{cases} \cos \theta (1-r) = \frac{t}{n} \cos \theta' \\ 1+r=t \end{cases} \Rightarrow \cos \theta - r \cos \theta = \frac{\cos \theta'}{n} + \frac{r \cos \theta'}{n} \Rightarrow$$

$$r = \frac{n \cos \theta - \cos \theta'}{n \cos \theta + \cos \theta'}, \quad t = 1+r = \frac{2n \cos \theta}{n \cos \theta + \cos \theta'}$$

The Fresnel coefficients for the E-field are similar to r and t above, except for a sign change in r and the elimination of n from the numerator of t .