Summer 2020 Written Comprehensive Exam
Opti 501

System of units: MKSA

1) A monochromatic plane-wave, having frequency ω and wave-vector $\boldsymbol{k}$, propagates in free space. For all practical purposes, one may assume that ω is a real-valued scalar, while $\boldsymbol{k}$ is a complex-valued vector, that is, $\boldsymbol{k} = \boldsymbol{k}' + \mathrm{i}\boldsymbol{k}''$. Let the scalar and vector potentials associated with this plane-wave be written as $\psi(\boldsymbol{r},t) = \psi_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ and $\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}_o \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$, respectively.

a) Write the differential equation relating the scalar and vector potentials in the Lorenz gauge, then derive the relation among ψ_o, $\boldsymbol{A}_o$, $\boldsymbol{k}$ and ω assuming the aforementioned plane-wave satisfies the Lorenz gauge.

b) Find expressions for the E- and B-fields of the plane-wave in terms of ψ_o, $\boldsymbol{A}_o$, $\boldsymbol{k}$ and ω.

c) Write the differential form of Maxwell's equations, then obtain the constraints on ψ_o, $\boldsymbol{A}_o$, $\boldsymbol{k}$ and ω that ensure the above plane-wave is a solution of Maxwell's equations.

d) Specify the condition(s) under which the plane-wave is evanescent (i.e., inhomogeneous).

e) Specify the condition(s) under which the plane-wave is homogeneous and linearly polarized.

f) Specify the condition(s) under which the plane-wave is homogeneous and circularly polarized.

g) In terms of $\boldsymbol{A}_o$, ψ_o, $\boldsymbol{k}$ and ω, find the time-averaged rate-of-flow of electromagnetic energy (per unit area per unit time) for the plane-wave.

Hint: You might find the following vector identities useful:

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$$

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2) A linearly-polarized light pulse of duration T, frequency ω_o, E-field amplitude E_0, and cross-sectional area A, propagates in free space. (The pulse is long enough and broad enough that one can ignore its spectral content and treat it simply as a section from a plane wave.) The pulse arrives at normal incidence at the flat surface of a linear, isotropic, homogeneous, semi-infinite material of refractive index n, where n is real-valued and greater than unity. You may assume that the transparent dielectric material is non-magnetic [i.e., $\mu(\omega)=1$] and non-dispersive (i.e., n does *not* vary with frequency ω within the bandwidth of the light pulse).

a) What is the total energy content of the light pulse in free space?

b) Describe the properties of the reflected light pulse (e.g., frequency, wavelength, duration, polarization state, total optical power).

c) Find the E-field and H-field amplitudes of the light pulse that enters the glass medium. Describe the properties of the light pulse that propagates within the glass medium.

d) Show that the total energy of the pulse is conserved upon reflection/transmission at the glass surface.

Solution to Problem 1)

a) Lorenz Gauge: $\boldsymbol{\nabla}\cdot\boldsymbol{A}+\dfrac{1}{c^2}\dfrac{\partial\psi}{\partial t}=0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}-(\mathrm{i}\omega/c^2)\psi_\mathrm{o}=0 \;\rightarrow\; \boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}=(\omega/c^2)\psi_\mathrm{o}.$

b) $\boldsymbol{E}=-\boldsymbol{\nabla}\psi-\dfrac{\partial\boldsymbol{A}}{\partial t} \;\rightarrow\; \boldsymbol{E}(\boldsymbol{r},t)=(-\mathrm{i}\boldsymbol{k}\psi_\mathrm{o}+\mathrm{i}\omega\boldsymbol{A}_\mathrm{o})\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$

$$\rightarrow\; \boldsymbol{E}_\mathrm{o}=\mathrm{i}(\omega\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}\psi_\mathrm{o}) \;\rightarrow\; \boldsymbol{E}_\mathrm{o}=\mathrm{i}\,\omega[\boldsymbol{A}_\mathrm{o}-(c/\omega)^2(\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o})\boldsymbol{k}].$$

$$\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A} \;\rightarrow\; \boldsymbol{B}(\boldsymbol{r},t)=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \;\rightarrow\; \boldsymbol{B}_\mathrm{o}=\mu_\mathrm{o}\boldsymbol{H}_\mathrm{o}=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o}.$$

c) In free space, $\rho_{\text{free}}=0$, $\boldsymbol{J}_{\text{free}}=0$, $\boldsymbol{P}=0$, and $\boldsymbol{M}=0$. Consequently, $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}$ and $\boldsymbol{B}=\mu_0\boldsymbol{H}$. Maxwell's equations thus become

i) $\boldsymbol{\nabla}\cdot\varepsilon_\mathrm{o}\boldsymbol{E}=0 \;\rightarrow\; \mathrm{i}\varepsilon_\mathrm{o}\boldsymbol{k}\cdot\boldsymbol{E}_\mathrm{o}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0 \;\rightarrow\; \boldsymbol{k}\cdot\boldsymbol{E}_\mathrm{o}=0 \;\rightarrow\; \boldsymbol{k}\cdot(\omega\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}\psi_\mathrm{o})=0$

$$\rightarrow\; \omega\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}^2\psi_\mathrm{o}=0 \;\rightarrow\; (\omega/c)^2\psi_\mathrm{o}-\boldsymbol{k}^2\psi_\mathrm{o}=0 \;\rightarrow\; \text{Either } \psi_o=0 \text{ or } \boldsymbol{k}^2=(\omega/c)^2.$$

ii) $\boldsymbol{\nabla}\times\boldsymbol{H}=\dfrac{\partial\varepsilon_\mathrm{o}\boldsymbol{E}}{\partial t} \;\rightarrow\; \boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{\nabla}\times\mu_\mathrm{o}\boldsymbol{H}=\dfrac{\partial\boldsymbol{E}}{c^2\partial t} \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o})=-(\mathrm{i}\omega/c^2)\boldsymbol{E}_\mathrm{o}$

$$\rightarrow\; (\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o})\boldsymbol{k}-\boldsymbol{k}^2\boldsymbol{A}_\mathrm{o}=-(\omega/c^2)(\omega\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}\psi_\mathrm{o})$$

$$\rightarrow\; [\boldsymbol{k}^2-(\omega/c)^2]\boldsymbol{A}_\mathrm{o}=[\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}-(\omega/c^2)\psi_\mathrm{o}]\boldsymbol{k}=0 \;\rightarrow\; \boldsymbol{k}^2=(\omega/c)^2.$$

iii) $\boldsymbol{\nabla}\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t} \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times[\mathrm{i}(\omega\boldsymbol{A}_\mathrm{o}-\boldsymbol{k}\psi_\mathrm{o})]=\mathrm{i}\omega(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o})$

$$\rightarrow\; \omega\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o}-(\boldsymbol{k}\times\boldsymbol{k})\psi_\mathrm{o}=\omega\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o} \;\rightarrow\; (\boldsymbol{k}\times\boldsymbol{k})\psi_\mathrm{o}=0 \;\rightarrow\; 0=0.$$

iv) $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\cdot(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\mathrm{o})=0 \;\rightarrow\; (\boldsymbol{k}\times\boldsymbol{k})\cdot\boldsymbol{A}_\mathrm{o}=0 \;\rightarrow\; 0=0.$

Thus, the only constraint on the parameters, aside from the Lorenz gauge $\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o}=(\omega/c^2)\psi_\mathrm{o}$, is $\boldsymbol{k}^2=(\omega/c)^2$.

d) The plane-wave is inhomogeneous (or evanescent) when the imaginary component of $\boldsymbol{k}$ is non-zero. The constraint $\boldsymbol{k}^2=(\omega/c)^2$ thus yields

$$\boldsymbol{k}^2=(\omega/c)^2 \;\rightarrow\; (\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')=(\omega/c)^2 \;\rightarrow\; \boldsymbol{k}'^2-\boldsymbol{k}''^2+2\mathrm{i}\,\boldsymbol{k}'\cdot\boldsymbol{k}''=(\omega/c)^2$$

$$\rightarrow\; \boldsymbol{k}'^2-\boldsymbol{k}''^2=(\omega/c)^2 \quad\text{and}\quad \boldsymbol{k}'\cdot\boldsymbol{k}''=0.$$

For the plane-wave to be evanescent, it is thus necessary for $\boldsymbol{k}'$ and $\boldsymbol{k}''$ to be orthogonal to each other. It is also necessary for $|\boldsymbol{k}'|$ to be greater than ω/c, so that $|\boldsymbol{k}''|$ will be real-valued.

e) When $\boldsymbol{k}$ is a real-valued vector, i.e., when $\boldsymbol{k}''=0$, the plane-wave will be homogeneous. Both $\boldsymbol{E}_\mathrm{o}$ and $\boldsymbol{B}_\mathrm{o}$ will then be proportional to the transverse component $\boldsymbol{A}_{\mathrm{o}\perp}=\boldsymbol{A}_\mathrm{o}-(c/\omega)^2(\boldsymbol{k}\cdot\boldsymbol{A}_\mathrm{o})\boldsymbol{k}$ of $\boldsymbol{A}_\mathrm{o}$, with $\boldsymbol{B}_\mathrm{o}$ rotated around the k-vector by 90º. The plane-wave will be linearly polarized if the real and imaginary parts of this transverse vector potential, namely, $\boldsymbol{A}'_{o\perp}$ and $\boldsymbol{A}''_{o\perp}$, happen to be parallel to each other, or if one of them (either $\boldsymbol{A}'_{o\perp}$ or $\boldsymbol{A}''_{o\perp}$) vanishes.

f) Again, the plane-wave is homogeneous when $\boldsymbol{k}''=0$. As before, $\boldsymbol{E}_o$ and $\boldsymbol{B}_o$ will be proportional to $\boldsymbol{A}_{o\perp}$, with $\boldsymbol{B}_o$ rotated around $\boldsymbol{k}$ by 90º. The plane-wave will be circularly polarized if $\boldsymbol{A}'_{o\perp}$ and $\boldsymbol{A}''_{o\perp}$ happen to be equal in magnitude and perpendicular to each other.

g) The time-averaged rate of flow of electromagnetic energy is given by the time-averaged Poynting vector, that is,

$$
\begin{aligned}
<\boldsymbol{S}(\boldsymbol{r},t)> &= \tfrac{1}{2}\operatorname{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \tfrac{1}{2}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\operatorname{Re}(\boldsymbol{E}_o\times\boldsymbol{H}_o^*)\\
&= \tfrac{1}{2}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\operatorname{Re}\{(\omega\boldsymbol{A}_o-\boldsymbol{k}\psi_o)\times\mu_o^{-1}\boldsymbol{k}^*\times\boldsymbol{A}_o^*\}\\
&= \frac{\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})}{2\mu_o}\operatorname{Re}\{[(\boldsymbol{A}_o\cdot\boldsymbol{A}_o^*)\boldsymbol{k}^*-(\boldsymbol{k}^*\cdot\boldsymbol{A}_o)\boldsymbol{A}_o^*]\omega+[(\boldsymbol{k}\cdot\boldsymbol{k}^*)\boldsymbol{A}_o^*-(\boldsymbol{k}\cdot\boldsymbol{A}_o^*)\boldsymbol{k}^*]\psi_o\}.
\end{aligned}
$$

Solution to Problem 2) a) The E-field energy-density is $\frac{1}{2}\varepsilon_0|\boldsymbol{E}|^2$. Since the E-field oscillates with frequency ω_0, time-averaging yields the average E-field energy-density as $\frac{1}{4}\varepsilon_o E_0^2$. Multiplying this into the volume cTA of the pulse, we obtain the E-field energy of the pulse as $\frac{1}{4}\varepsilon_o cTAE_0^2$. Similarly, the amplitude of the H-field of the light is $H_0 = E_0/Z_o$. Since the time-averaged magnetic energy density in vacuum is given by $\frac{1}{4}\mu_o H_0^2 = \frac{1}{4}\varepsilon_o E_0^2$, the magnetic energy of the pulse is equal to its electric energy. The total energy is thus given by $\frac{1}{2}\varepsilon_o cTAE_0^2$.

Alternatively, we may compute the time-averaged Poynting vector as follows:

$$<\boldsymbol{S}> = \tfrac{1}{2}\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \tfrac{1}{2}E_0 H_0\hat{\boldsymbol{z}} = [E_0^2/(2Z_o)]\hat{\boldsymbol{z}}.$$

This is the rate of flow of energy per unit area per unit time at any given cross-section of the light pulse. Multiplication with A and T then yields the total energy of the pulse as $ATE_0^2/(2Z_o)$. Considering that $\varepsilon_0 c = 1/Z_0$, the two expressions obtained above for the total pulse energy are exactly the same.

b) The reflected pulse has the same frequency ω_0 and the same wavelength $\lambda_0 = 2\pi c/\omega_0$ as the incident pulse. Its polarization state is also linear and in the same direction as the incident polarization. The pulse duration and cross-sectional area remain T and A, respectively. The only things that change are the field amplitudes E_0 and H_0, which are multiplied by the Fresnel reflection coefficient $\rho = (1-n)/(1+n)$. The reflected pulse energy is therefore given by ρ^2 times the incident pulse energy, that is, $(1-n)^2 ATE_0^2/[2Z_o(1+n)^2]$.

c) The Fresnel transmission coefficient at the entrance facet of the glass slab is $\tau = 1+\rho = 2/(1+n)$. This means that the E-field amplitude inside the glass slab is $2E_0/(1+n)$. The H-field amplitude is n times the E-field amplitude divided by Z_0, that is, $H_0 = 2nE_0/[Z_0(1+n)]$. Therefore, the z-component of the Poynting vector inside the slab is $<S_z> = 2nE_0^2/[Z_o(1+n)^2]$. Since the pulse duration T and the cross-sectional area A inside the slab remain the same as outside, the total energy of the transmitted pulse is $2nATE_0^2/[Z_o(1+n)^2]$. Other properties of the transmitted pulse are: frequency $= \omega_0$, wavelength $\lambda = \lambda_0/n$, pulse length $= cT/n$, polarization state = linear and in the same direction as the incident pulse.

Alternatively, one may evaluate the energy densities of the E and H fields separately, then add them together. We find

$$\text{Time-averaged } E\text{-field energy density} = \tfrac{1}{4}\varepsilon_o\varepsilon(\tau E_0)^2 = \tfrac{1}{4}\varepsilon_o n^2[2E_0/(1+n)]^2 = \varepsilon_o n^2 E_0^2/(1+n)^2.$$

$$\text{Time-averaged } H\text{-field energy density} = \tfrac{1}{4}\mu_o(n\tau E_0/Z_o)^2 = \tfrac{1}{4}\mu_o\{2nE_0/[Z_o(1+n)]\}^2 = \varepsilon_o n^2 E_0^2/(1+n)^2.$$

Adding the above energy densities, then multiplying by the pulse volume cAT/n yields the same result as before, namely, transmitted pulse energy $= 2\varepsilon_o cnATE_0^2/(1+n)^2$.

d) Reflected plus transmitted pulse energy =

$$(1-n)^2 ATE_0^2/[2Z_o(1+n)^2] + 2nATE_0^2/[Z_o(1+n)^2] = ATE_0^2/(2Z_o).$$

This is the same as the incident pulse energy; therefore, energy is conserved.