Fall 2012 Written Comprehensive Exam
Opti 501

System of units: MKSA

A monochromatic plane-wave, having frequency ω and wave-vector $\boldsymbol{k}$, propagates in free space. For all practical purposes, one may assume that ω is a real-valued scalar, while $\boldsymbol{k}$ is a complex-valued vector, that is, $\boldsymbol{k} = \boldsymbol{k}' + \mathrm{i}\boldsymbol{k}''$. Let the scalar and vector potentials associated with this plane-wave be written as $\psi(\boldsymbol{r},t) = \psi_\mathrm{o} \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ and $\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}_\mathrm{o} \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$, respectively.

2 Pts a) Write the differential equation relating the scalar and vector potentials in the Lorenz gauge, then derive the relation among ψ_o, $\boldsymbol{A}_\mathrm{o}$, $\boldsymbol{k}$ and ω assuming the aforementioned plane-wave satisfies the Lorenz gauge.

1 Pt b) Find expressions for the E- and B-fields of the plane-wave in terms of ψ_o, $\boldsymbol{A}_\mathrm{o}$, $\boldsymbol{k}$ and ω.

3 Pts c) Write the differential form of Maxwell's equations, then obtain the constraints on ψ_o, $\boldsymbol{A}_\mathrm{o}$, $\boldsymbol{k}$ and ω that ensure the above plane-wave is a solution of Maxwell's equations.

1 Pt d) Specify the condition(s) under which the plane-wave is evanescent (i.e., inhomogeneous).

1 Pt e) Specify the condition(s) under which the plane-wave is homogeneous and linearly polarized.

1 Pt f) Specify the condition(s) under which the plane-wave is homogeneous and circularly polarized.

1 Pt g) In terms of $\boldsymbol{A}_\mathrm{o}$, ψ_o, $\boldsymbol{k}$ and ω, find the time-averaged rate-of-flow of electromagnetic energy (per unit area per unit time) for the plane-wave.

Hint: You might find the following vector identities useful:

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$$

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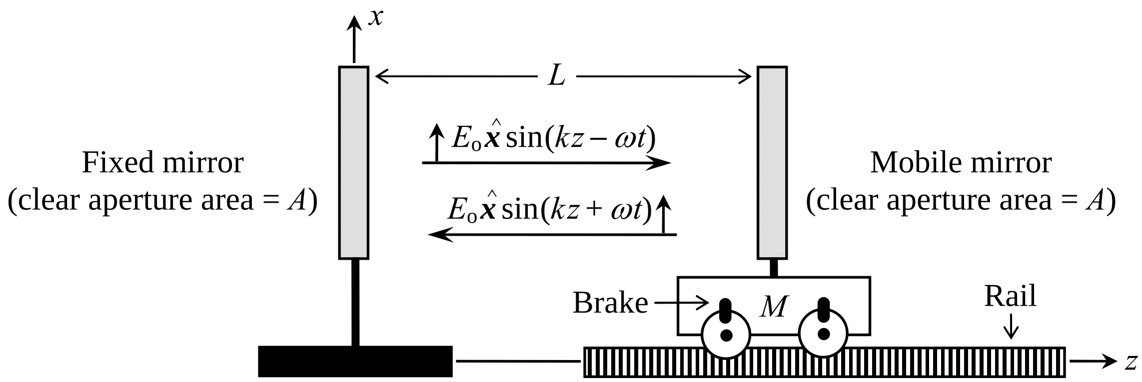
A pair of monochromatic, linearly-polarized, counter-propagating plane waves of frequency ω is trapped in free space between two perfectly electrically conducting flat mirrors, as shown in the figure. The distance between the mirrors is $L = N\lambda_o/2$, where $\lambda_o = 2\pi c/\omega$ is the vacuum wavelength, and N is an arbitrary integer. The E-fields of the two plane-waves are given by $E_o\hat{\boldsymbol{x}}\sin(kz \pm \omega t)$, where $k = \omega/c$. The cross-sectional area A of the two beams may be assumed to be large and uniform throughout the cavity.



2 Pts a) Write expressions for the total E- and H-fields as functions of z and t within the cavity.

2 Pts b) Find the total electromagnetic energy contained within the volume AL of the cavity. Considering that each photon is known to have an energy $\hbar\omega$, where $\hbar$ is Planck's reduced constant, how many photons are trapped inside the cavity?

2 Pts c) Find the induced current-density $\boldsymbol{J}_s(t)$ at the surface of each mirror, then use this current-density to determine the Lorentz force exerted by the trapped radiation on each mirror.

At $t = t_o$ the brakes preventing the motion of the mirror located at $z = L$ are released. Thereafter, radiation pressure pushes this mirror forward along the rail, with negligible losses due to friction. A short time later, at $t = t_o + \Delta t$, where $\Delta t = 2L/c$, the mirror will be at $z = L + \Delta z$ and will have acquired a velocity $V\hat{\boldsymbol{z}}$. Assume the mass M of the mirror is sufficiently large that relativistic effects may be ignored.

1 Pt d) Find the kinetic energy $\frac{1}{2}MV^2$ of the mirror by calculating the mechanical work done by the radiation pressure in moving the mirror from $z = L$ to $z = L + \Delta z$.

1 Pt e) Find the electromagnetic momentum transferred to the mirror during the interval $\Delta t = 2L/c$.

1 Pt f) Assuming that the number N of half-wavelengths that fit within the length of the cavity remains unchanged, find the relative shift in frequency, $\Delta\omega/\omega$, as a result of the expansion of the cavity from L to $L + \Delta z$.

1 Pt g) Combining the results obtained in parts (d) – (f), find the relation between the frequency shift $\Delta\omega$ and the acquired mirror velocity V at $t = t_o + \Delta t$.

Hint: $\sin(a) + \sin(b) = 2\sin[\frac{1}{2}(a+b)]\cos[\frac{1}{2}(a-b)]$ and $\sin(a) - \sin(b) = 2\sin[\frac{1}{2}(a-b)]\cos[\frac{1}{2}(a+b)]$.