Opti 501 **Midterm Exam** (11/13/2008) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

---

1. The vector and scalar potentials $\boldsymbol{A}(\boldsymbol{r},t)$ and $\psi(\boldsymbol{r},t)$ are obtained from the current and charge density distributions $\boldsymbol{J}(\boldsymbol{r},t)$ and $\rho(\boldsymbol{r},t)$ following standard procedure. Accordingly, the potential functions satisfy the Lorentz gauge $\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t)+(1/c^2)\partial\psi/\partial t=0$. Let $f(\boldsymbol{r},t)$ be an arbitrary, real-valued, scalar function defined over the relevant region of space-time, then define new potential functions $\boldsymbol{A}'(\boldsymbol{r},t)=\boldsymbol{A}(\boldsymbol{r},t)+\nabla f(\boldsymbol{r},t)$ and $\psi'(\boldsymbol{r},t)=\psi(\boldsymbol{r},t)-\partial f(\boldsymbol{r},t)/\partial t$.

1 pt a) Show that the magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$ derived from the new vector potential $\boldsymbol{A}'(\boldsymbol{r},t)$ is the same as that obtained from the old potential $\boldsymbol{A}(\boldsymbol{r},t)$.

1 pt b) Show that the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ derived from the new potentials $\boldsymbol{A}'(\boldsymbol{r},t)$ and $\psi'(\boldsymbol{r},t)$ is the same as that obtained from the old potentials $\boldsymbol{A}(\boldsymbol{r},t)$ and $\psi(\boldsymbol{r},t)$.

2 pts c) What restrictions should one impose on $f(\boldsymbol{r},t)$ to ensure that the new potentials also satisfy the Lorentz gauge?

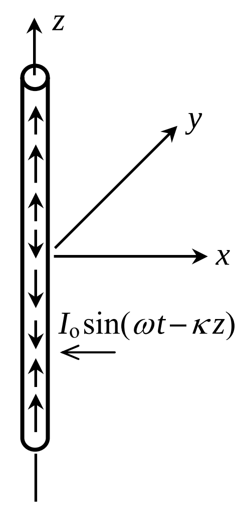
2 pts d) Under what circumstances will the new potentials satisfy the Coulomb gauge $\nabla\cdot\boldsymbol{A}'(\boldsymbol{r},t)=0$? In Coulomb gauge, write differential equations that relate $\psi'(\boldsymbol{r},t)$ and $\boldsymbol{A}'(\boldsymbol{r},t)$ to $\rho(\boldsymbol{r},t)$ and $\boldsymbol{J}(\boldsymbol{r},t)$.

---

2. An infinitely-long thin wire carries a current $I(z,t)=I_o\sin(\omega t-\kappa z)$ along the z-axis. For $0<\kappa<k_o$, where $k_o=\omega/c$, the scalar and vector potentials in cylindrical coordinates are

$$\psi(\rho,z,t)=-\frac{\kappa I_o}{4\varepsilon_o\omega}\left[Y_o(\sqrt{k_o^2-\kappa^2}\,\rho)\sin(\omega t-\kappa z)+J_o(\sqrt{k_o^2-\kappa^2}\,\rho)\cos(\omega t-\kappa z)\right],$$

$$\boldsymbol{A}(\rho,z,t)=-\frac{\mu_o I_o\hat{\boldsymbol{z}}}{4}\left[Y_o(\sqrt{k_o^2-\kappa^2}\,\rho)\sin(\omega t-\kappa z)+J_o(\sqrt{k_o^2-\kappa^2}\,\rho)\cos(\omega t-\kappa z)\right].$$



1 pt a) Use the continuity equation to find the charge distribution along the length of the wire.

1 pt b) Show that $\boldsymbol{A}(\rho,z,t)$ and $\psi(\rho,z,t)$ satisfy the Lorentz gauge.

2 pts c) Find the E- and H-fields in the region outside the wire.

2 pts d) Using approximate expressions for the Bessel functions, find the Poynting vector $\boldsymbol{S}(\rho,z,t)$ in the far-field region.

---

3. In empty free space where $\rho_{\text{free}}=0$, $\boldsymbol{J}_{\text{free}}=0$, $\boldsymbol{P}=0$, $\boldsymbol{M}=0$, a plane-wave has potential functions $\boldsymbol{A}(\boldsymbol{r},t)=\boldsymbol{A}_o\exp[\mathrm{i}(k_o\boldsymbol{\sigma}\cdot\boldsymbol{r}-\omega t)]$ and $\psi(\boldsymbol{r},t)=\psi_o\exp[\mathrm{i}(k_o\boldsymbol{\sigma}\cdot\boldsymbol{r}-\omega t)]$. Here $k_o=\omega/c$, $\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}=1$, and the potentials satisfy the Lorentz gauge $\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t)+(1/c^2)\partial\psi/\partial t=0$.

2 pts a) What constraint does the Lorentz gauge impose on $\boldsymbol{A}_o$, ψ_o, and $\boldsymbol{\sigma}$?

2 pts b) Derive expressions for the plane-wave's E-field and H-field in terms of the defining parameters of the potential functions.

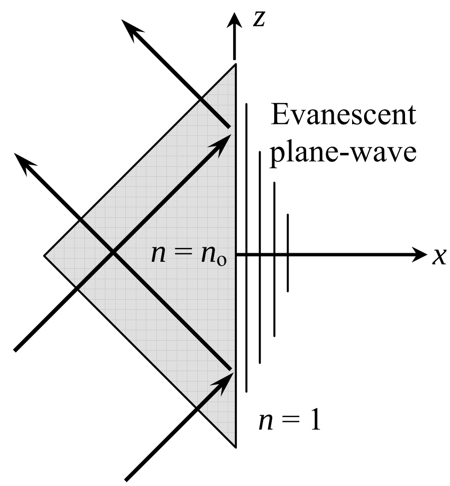
2 pts c) Show that the E- and H-fields obtained in part (b) satisfy all four Maxwell's equations.

---

4. A homogeneous plane-wave undergoes total internal reflection within a glass prism of refractive index $n = n_o$, as shown. The resulting evanescent plane-wave in the free-space region beyond the prism (i.e., $x \geq 0$) has the following E- and H-fields:

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_o \exp[\mathrm{i}(k_o \boldsymbol{\sigma} \cdot \boldsymbol{r} - \omega t)],$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_o \exp[\mathrm{i}(k_o \boldsymbol{\sigma} \cdot \boldsymbol{r} - \omega t)].$$



Let $\boldsymbol{\sigma} = \mathrm{i}\sigma_x \hat{\boldsymbol{x}} + \sigma_z \hat{\boldsymbol{z}}$, while $\boldsymbol{E}_o = E_{xo}\hat{\boldsymbol{x}} + E_{yo}\hat{\boldsymbol{y}} + E_{zo}\hat{\boldsymbol{z}}$ and $\boldsymbol{H}_o = H_{xo}\hat{\boldsymbol{x}} + H_{yo}\hat{\boldsymbol{y}} + H_{zo}\hat{\boldsymbol{z}}$. In general, σ_x and σ_z are real-valued, while the components of $\boldsymbol{E}_o$ and $\boldsymbol{H}_o$ are complex.

1 pt a) What is the relationship between σ_x and σ_z?

1 pt b) What does Maxwell's first equation say about the relation between E_{xo} and E_{zo}?

1 pt c) What does Maxwell's fourth equation say about the relation between H_{xo} and H_{zo}?

2 pts d) What is the relation between (E_{xo}, E_{yo}, E_{zo}) and (H_{xo}, H_{yo}, H_{zo}) based on Maxwell's third equation?

1 pt e) If $E_{zo} = 0$, which components of $\boldsymbol{E}_o$ and $\boldsymbol{H}_o$ will vanish as well?

1 pt f) If $H_{zo} = 0$, which components of $\boldsymbol{E}_o$ and $\boldsymbol{H}_o$ will vanish as well?
