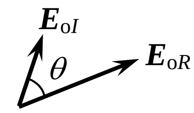
Opti 501 **Midterm Exam** (11/13/2007) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.



1. The complex E-field amplitude associated with a plane-wave may generally be written as $\boldsymbol{E}_o = \boldsymbol{E}_{oR} + \mathrm{i}\,\boldsymbol{E}_{oI}$. The time-dependent E-field at a point $\boldsymbol{r}_o$ in space (ignoring a time-independent amplitude and phase factor), is then written $\boldsymbol{E}(\boldsymbol{r}_o, t) = \boldsymbol{E}_{oR}\cos(\omega t) + \boldsymbol{E}_{oI}\sin(\omega t)$. In general, the tip of this E-field vector traces an ellipse during each oscillation period.

1 pt a) Can a plane-wave be linearly-polarized if the vectors $\boldsymbol{E}_{oR}$ and $\boldsymbol{E}_{oI}$ have different directions? Explain.

1 pt b) If $\boldsymbol{E}_{oR}$ and $\boldsymbol{E}_{oI}$ are aligned (i.e., they are in the same direction), what is the E-field magnitude?

2 pts c) Can a plane-wave be circularly-polarized if $\boldsymbol{E}_{oR}$ and $\boldsymbol{E}_{oI}$ happen to have different lengths, i.e., if $|\boldsymbol{E}_{oR}| \neq |\boldsymbol{E}_{oI}|$? Explain.

2 pts d) Assuming that $|\boldsymbol{E}_{oR}| = |\boldsymbol{E}_{oI}|$, can the plane-wave be circularly-polarized if $\boldsymbol{E}_{oR}$ is *not* orthogonal to $\boldsymbol{E}_{oI}$ (that is, if $\theta \neq 90°$)? Explain.

2 pts e) For a homogeneous plane-wave in free space, the propagation vector $\boldsymbol{\sigma} = \boldsymbol{\sigma}_R + \mathrm{i}\boldsymbol{\sigma}_I$ has the property that $|\boldsymbol{\sigma}_R| = 1.0$ and $\boldsymbol{\sigma}_I = 0$. Express the magnetic field vector $\boldsymbol{H}_o = \boldsymbol{H}_{oR} + \mathrm{i}\,\boldsymbol{H}_{oI}$ in terms of $\boldsymbol{\sigma}_R$, $\boldsymbol{E}_{oR}$, $\boldsymbol{E}_{oI}$, and the free-space impedance Z_o.

2 pts f) For a homogeneous plane-wave in free space, write the Poynting vector $\boldsymbol{S}(\boldsymbol{r}, t)$ in terms of $\boldsymbol{E}_{oR}$, $\boldsymbol{E}_{oI}$, $\boldsymbol{\sigma}_R$, Z_o, ω and $k_o = \omega/c$. Explore the behavior of $\boldsymbol{S}(\boldsymbol{r}, t)$ for circularly-polarized light.

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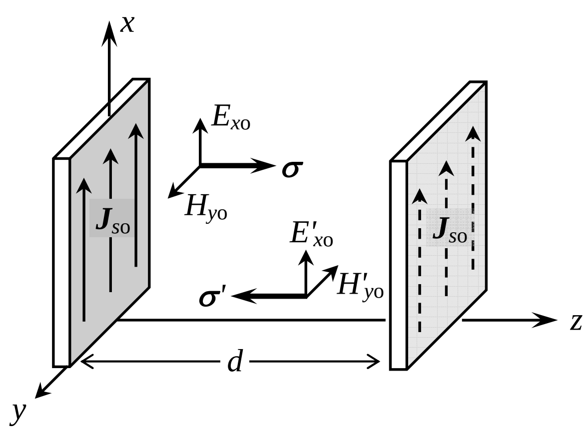
2. Two counter-propagating, linearly-polarized, homogeneous plane waves, both having frequency ω and wavelength λ_o, are trapped in the free-space region between a pair of perfectly conducting mirrors, as shown. The mirrors are parallel to the xy-plane, their separation d being an integer-multiple of $\lambda_o/2$. The propagation directions are $\boldsymbol{\sigma} = \hat{\boldsymbol{z}}$ and $\boldsymbol{\sigma}' = -\hat{\boldsymbol{z}}$. The (complex) E-field amplitudes $E_{xo} = |E_{xo}|\exp(\mathrm{i}\phi_o)$ and $E'_{xo} = |E'_{xo}|\exp(\mathrm{i}\phi'_o)$ have equal magnitudes, namely, $|E_{xo}| = |E'_{xo}|$.



3 pts a) Write expressions for the total E- and H-fields in the cavity between the mirrors.

2 pts b) In terms of $|E_{xo}|$, find the magnitude of the surface current density $J_{so}\hat{\boldsymbol{x}}$ on the mirror surfaces.

3 pts c) Determine the total E- and H-field energies trapped within the cavity. Show that, at those instants of time when the E-field energy is zero, the H-field energy is at a maximum, and vice-versa.

2 pts d) Write an expression for the Poynting vector $\boldsymbol{S}(z, t)$ inside the cavity. Interpret the oscillations of $\boldsymbol{S}(z, t)$ at a fixed location in space as time varies during each oscillation period.

Hint: $\cos(a) + \cos(b) = 2\cos[½(a+b)]\cos[½(a-b)]$; $\cos(a) - \cos(b) = -2\sin[½(a+b)]\sin[½(a-b)]$.

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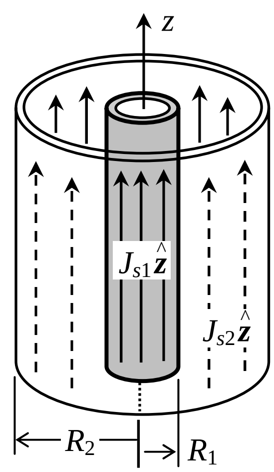
3. For an infinitely long, hollow cylinder of radius R carrying a uniform current $I(t) = I_o \sin(\omega t)$ on its exterior surface, we have shown that the E- and H-fields inside and outside the cylinder are given by

$$\boldsymbol{E}(\boldsymbol{r}, t) = \tfrac{1}{4} Z_o I_o k_o \begin{cases} J_o(k_o R)\big[Y_o(k_o\rho)\cos(\omega t) - J_o(k_o\rho)\sin(\omega t)\big]\hat{\boldsymbol{z}}; & \rho \geq R, \\ \big[Y_o(k_o R)\cos(\omega t) - J_o(k_o R)\sin(\omega t)\big] J_o(k_o\rho)\hat{\boldsymbol{z}}; & \rho \leq R. \end{cases}$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = -\tfrac{1}{4} I_o k_o \begin{cases} J_o(k_o R)\big[Y_1(k_o\rho)\sin(\omega t) + J_1(k_o\rho)\cos(\omega t)\big]\hat{\boldsymbol{\phi}}; & \rho > R, \\ \big[Y_o(k_o R)\sin(\omega t) + J_o(k_o R)\cos(\omega t)\big] J_1(k_o\rho)\hat{\boldsymbol{\phi}}; & \rho < R. \end{cases}$$

In the present problem we are interested in the electromagnetic field trapped between two concentric, **perfectly conducting**, infinitely long, hollow cylinders of radii R_1 and R_2, as shown. The small cylinder of radius R_1 carries a uniform current $I_1(t) = I_{1o}\sin(\omega t)$ on its exterior surface along the z-axis, corresponding to the surface current density $J_{s1}(t)\hat{\boldsymbol{z}}$. The large cylinder of radius R_2 carries a uniform current $I_2(t) = I_{2o}\sin(\omega t)$ on its interior surface, also along the z-axis, corresponding to the surface current density $J_{s2}(t)\hat{\boldsymbol{z}}$.



3 pts a) Write expressions for the total E-field and H-field in the cavity formed between the two cylinders. Use the boundary conditions at the outer surface of the small cylinder, as well as those at the inner surface of the large cylinder, to find the required condition for the existence of trapped fields within the cavity.

1 pt b) Determine the E- and H-fields inside the small cylinder.

1 pt c) Determine the E- and H-fields outside the large cylinder.