Opti 501 **Midterm Exam** (11/14/2006) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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5 pts 1. Consider a region of free-space containing the static (i.e., time-independent) electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$, respectively. The region may contain stationary charges and time-independent currents, but the current density $\boldsymbol{J}(\boldsymbol{r})$, where present, is orthogonal to the local E-field, that is, $\boldsymbol{E}(\boldsymbol{r})\cdot\boldsymbol{J}(\boldsymbol{r})=0$. Prove that, within this region of space, $\nabla\cdot\boldsymbol{S}(\boldsymbol{r})=0$, where $\boldsymbol{S}(\boldsymbol{r})=\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}(\boldsymbol{r})$ is the Poynting vector distribution throughout the region.

Hint: $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A})-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$

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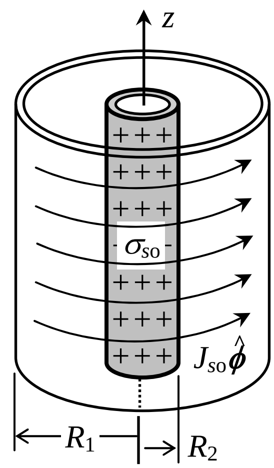
2. An infinitely-long, thin solenoid of radius R_1 carries a uniform, time-independent surface current density $J_{so}\hat{\boldsymbol{\phi}}$. Inside the solenoid and sharing the axis with it, is another infinitely-long, thin, hollow cylinder of radius R_2. The surface of the small cylinder is uniformly charged with a time-independent charge density σ_{so}. (This will induce negative charges on the inside wall of the solenoid, but the outer wall of the solenoid remains free of charge.)



2 pts a) What is the magnetic field distribution $\boldsymbol{H}(\boldsymbol{r})$ throughout the entire space?
3 pts b) What is the electric field distribution $\boldsymbol{E}(\boldsymbol{r})$ throughout the entire space?
3 pts c) Find the distribution of the Poynting vector $\boldsymbol{S}(\boldsymbol{r})$ inside and outside the cylinders.
2 pts d) Determine the divergence of $\boldsymbol{S}(\boldsymbol{r})$, and show that the integral of $\boldsymbol{S}(\boldsymbol{r})$ over any closed surface in this system is zero.

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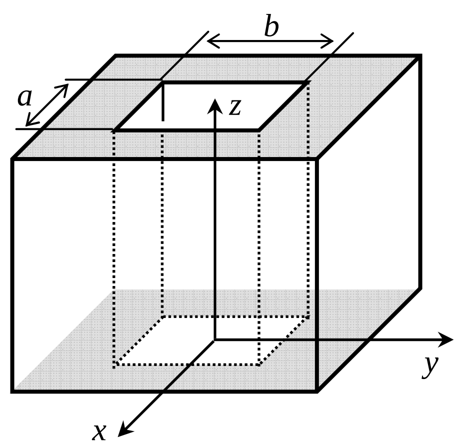
3. The hollow, rectangular core of a perfect metallic conductor is used as a waveguide. The core's dimensions are $a\times b$, as shown, and the propagation direction is along the z-axis. The guided mode is the superposition of four ***homogeneous*** plane-waves of the form $\boldsymbol{E}_o \exp[i(k_o\boldsymbol{\sigma}.\,\boldsymbol{r}-\omega t)]$, specified as follows:



$\boldsymbol{\sigma}_1=(\sigma_x,\sigma_y,\sigma_z)$, $\boldsymbol{E}_{o1}=(E_{ox},E_{oy},E_{oz})$, $\boldsymbol{H}_{o1}=(H_{ox},H_{oy},H_{oz})$.
$\boldsymbol{\sigma}_2=(-\sigma_x,\sigma_y,\sigma_z)$, $\boldsymbol{E}_{o2}=(-E_{ox},E_{oy},E_{oz})$, $\boldsymbol{H}_{o2}=(H_{ox},-H_{oy},-H_{oz})$.
$\boldsymbol{\sigma}_3=(\sigma_x,-\sigma_y,\sigma_z)$, $\boldsymbol{E}_{o3}=(E_{ox},-E_{oy},E_{oz})$, $\boldsymbol{H}_{o3}=(-H_{ox},H_{oy},-H_{oz})$.
$\boldsymbol{\sigma}_4=(-\sigma_x,-\sigma_y,\sigma_z)$, $\boldsymbol{E}_{o4}=(-E_{ox},-E_{oy},E_{oz})$, $\boldsymbol{H}_{o4}=(-H_{ox},-H_{oy},H_{oz})$.

The signs of the various components are chosen such that the constraints $\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}=1$, $\boldsymbol{\sigma}\cdot\boldsymbol{E}_o=0$, $\boldsymbol{\sigma}\cdot\boldsymbol{H}_o=0$, and $Z_o\boldsymbol{H}_o=\boldsymbol{\sigma}\times\boldsymbol{E}_o$, if satisfied for one plane-wave, will be satisfied for all.

5 pts a) Determine the values of σ_x, σ_y as functions of a, b, and λ_o, so that the tangential E-field and perpendicular H-field components vanish everywhere on the inner walls of the waveguide.
5 pts b) Find the distributions of surface charge density σ_s and surface current density $\boldsymbol{J}_s$ on the inner walls, and show that the conservation of charge equation, $\nabla\cdot\boldsymbol{J}_s+\partial\sigma_s/\partial t=0$, is satisfied.
