Opti 501 **Midterm Exam** (10/8/2009) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1. Let the density of free charge, distributed in free space (i.e., vacuum), be given by an arbitrary, real-valued function $\rho_{\text{free}}(\boldsymbol{r},t)$, where $\boldsymbol{r}$ specifies position in 3D space while t indicates an instant of time. Suppose a vector field $\boldsymbol{V}(\boldsymbol{r},t)$ can be assigned to this charge distribution so that the charges located at $\boldsymbol{r}$ can be said to have a well-defined, real-valued velocity $\boldsymbol{V}(\boldsymbol{r},t)$ at time t.

1 pt a) What is the free-current density $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$ in the above system?

1 pt b) Verify that $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$ has the correct dimensionality.

2 pt c) Write the charge/current continuity equation in both differential and integral forms. Explain the physical meaning of this equation by taking an arbitrary volume of space and observing the behavior of the total charge $Q_{\text{free}}(t) = \int \rho_{\text{free}}(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r}$ within this volume during a short time interval Δt.

2 pts d) Starting with the differential form of the continuity equation, apply the Fourier transform operation to both sides to derive the continuity equation in the Fourier domain.

Problem 2. Two species of charged particles (e.g., positive and negative ions within an electrolytic solution) reside in the same region of space and time. The corresponding charge densities are $\rho_1(\boldsymbol{r},t)$ and $\rho_2(\boldsymbol{r},t)$, while the corresponding velocity fields are $\boldsymbol{V}_1(\boldsymbol{r},t)$ and $\boldsymbol{V}_2(\boldsymbol{r},t)$.

1 pt a) What is the total charge density of the above system?

2 pts b) What is the total current density of the above system?

2 pts c) Given the possibility that opposite and equal charges can annihilate each other (e.g., in a salt solution, the Na^+ and Cl^- ions could combine to form neutral NaCl molecules, or, in vacuum, individual electrons and positrons could collide and turn into neutral photons), can one still write the charge/current continuity equation for each of the charged species 1 and 2? Explain.

1 pt d) Can one write the charge/current continuity equation for the combined system? Explain.

Problem 3. A region of space-time contains a material medium with the polarization density distribution $\boldsymbol{P}(\boldsymbol{r},t)$. No other sources of the electromagnetic field reside within the region of interest, that is, $\rho_{\text{free}}(\boldsymbol{r},t)=0$, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)=0$ and $\boldsymbol{M}(\boldsymbol{r},t)=0$. The electric and magnetic fields inside the medium are $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$, respectively.

1 pt a) What are the total charge-density and current-density distributions within the above system?

2 pts b) Let $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r})\cos(\omega_o t)$, where $\boldsymbol{E}(\boldsymbol{r})$ is a real-valued vector function of $\boldsymbol{r}$, and ω_o is a fixed, real-valued frequency. Suppose the polarization density is related to the local electric field in the following way: $\boldsymbol{P}(\boldsymbol{r},t)=\varepsilon_o\chi_o\boldsymbol{E}(\boldsymbol{r})\cos(\omega_o t-\phi_o)$. Here ε_o is the usual permittivity of free space, and $\varepsilon_o\chi_o\exp(\mathrm{i}\phi_o)$, the so-called dielectric susceptibility of the material medium, is a complex-valued constant, independent of $\boldsymbol{r}$ and t. (Note that both χ_o and ϕ_o are real-valued.)

Under what circumstances does the current density $\boldsymbol{J}(\boldsymbol{r},t)$ oscillate in-phase with the local electric field $\boldsymbol{E}(\boldsymbol{r},t)$? (**Note**: "in-phase" means no phase difference between the two sinusoidal oscillations.)

2 pts c) In general, a monochromatic (i.e., single-frequency) electric field is written as follows: $\boldsymbol{E}(\boldsymbol{r},t) = \text{Real}\{[\boldsymbol{E}'(\boldsymbol{r}) + \text{i}\boldsymbol{E}''(\boldsymbol{r})]\exp(-\text{i}\omega_\text{o}t)\}$, where both $\boldsymbol{E}'(\boldsymbol{r})$ and $\boldsymbol{E}''(\boldsymbol{r})$ are real-valued vector functions of $\boldsymbol{r}$. Assuming the material medium is homogeneous, isotropic and linear, with a dielectric susceptibility $\varepsilon_\text{o}\chi_\text{o}\exp(\text{i}\phi_\text{o})$, write expressions for $\boldsymbol{P}(\boldsymbol{r},t)$, $\boldsymbol{J}(\boldsymbol{r},t)$, and the displacement field $\boldsymbol{D}(\boldsymbol{r},t)$. [You may use the complex-valued relative permittivity $\varepsilon\exp(\text{i}\eta) = 1 + \chi_\text{o}\exp(\text{i}\phi_\text{o})$ to simplify the expression for $\boldsymbol{D}(\boldsymbol{r},t)$.]

1 pt d) Within the material medium described in part (c) above, determine the values of $\nabla\cdot\boldsymbol{D}(\boldsymbol{r},t)$, $\nabla\cdot\boldsymbol{E}(\boldsymbol{r},t)$ and $\nabla\cdot\boldsymbol{P}(\boldsymbol{r},t)$.

Problem 4. A magnetized disk with a central hole has radii R_1, R_2 and height h, as shown in Fig. (a) below. The disk rotates around the z-axis at a constant angular velocity $\omega_\text{o} = 2\pi f_\text{o}$. The magnetization distribution within the disk is given by $\boldsymbol{M}(\boldsymbol{r},t) = (R_1/r)M_\text{o}\hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is the unit vector along the radial direction in a cylindrical coordinate system.

2 pts a) Using the standard functions $\text{Circ}(r)$ and $\text{Rect}(z)$, express the magnetization distribution within the cylindrical coordinate system.

2 pts b) Determine the bound electrical current density $\boldsymbol{J}_\text{b}^{(\text{e})}(\boldsymbol{r},t)$ and the bound magnetic monopole density $\rho_\text{b}^{(\text{m})}(\boldsymbol{r},t)$ of the magnetized disk.

1 pt c) What is the magnetic monopole current density on the inner and outer surfaces of the cylindrical disk?

2 pts d) The magnetic monopole current density determined in part (c) above can be said to arise from a uniform polarization density $\boldsymbol{P}(\boldsymbol{r},t) = P_\text{o}\hat{\boldsymbol{z}}$ within the disk, as shown in Fig. (b). What value of P_o would yield the same equivalent magnetic monopole current density as in part (c)?