Opti 501 **Midterm Exam** (10/10/2006) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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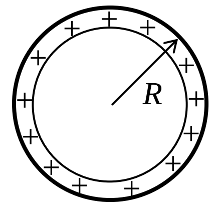
1. Consider a spherical shell of radius R and negligible thickness, uniformly charged with a surface charge density σ_o.



2 pts a) Find the E-field distribution both inside and outside the shell.

2 pts b) Determine the total field energy by integrating the E-field's energy density over the entire space.

2 pts c) Find the effective field acting on the surface charges, then compute the work done by this field on the shell when shrinking the shell by reducing its radius from R to $R - \Delta R$. (ΔR signifies a small change in the radius of the sphere.)

2 pts d) Show that the external work done on the sphere (while shrinking it) is equal to the E-field energy that now resides in the thin shell located between $R - \Delta R$ and R.

Hint: Express your results in terms of the total charge Q of the shell ($Q = 4\pi R^2 \sigma_o$).

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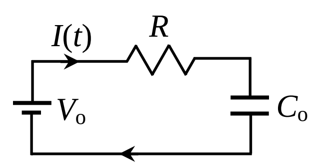
2. A capacitor C_o is connected via a series resistor R to a constant-voltage battery V_o. The capacitor plates have area A_o and separation d_o, so that $C_o = \varepsilon_o A_o/d_o$. The circuit has reached the steady-state in which the current is zero, the capacitor plates have total charge $Q_o = C_o V_o$, and the E-field between the plates is $E_o = V_o/d_o$. The total E-field energy stored in the capacitor is thus $W_o = ½\varepsilon_o E_o^2 A_o d_o = ½C_o V_o^2$.



Suppose that at $t = 0$ the capacitor plates are suddenly brought closer together, by shrinking the distance between the plates from d_o to d_1. (The movement is rapid enough that it may be assumed instantaneous.) At the end of the movement, namely, at $t = 0^+$, the capacitance is $C_1 = \varepsilon_o A_o/d_1$, but Q_o has not had time to change and, therefore, the capacitor's voltage has dropped from V_o (at $t = 0^-$) to $V_1(t = 0^+) = Q_o/C_1 = (C_o/C_1)V_o$.

2 pts a) Show that, for $t > 0^+$, the current in the circuit is $I(t) = [(C_1 - C_o)V_o/(RC_1)] \exp(-t/RC_1)$.

2 pts b) Determine the total energy delivered by the battery to the circuit after the capacitor changes from C_o to C_1.

2 pts c) Determine the total energy consumed in the resistor R after the change of the capacitor.

2 pts d) How much mechanical work is expended when the distance between the capacitor plates is reduced from d_o to d_1?

1 pt e) Show that the energy delivered to the capacitor by the battery, minus the mechanical work performed by the capacitor on the outside world, accounts for the change in the stored E-field energy of the capacitor from $W_o = ½C_o V_o^2$ at $t = 0^-$ to $W_1 = ½C_1 V_o^2$ at $t = \infty$.

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3. Figure 3(a) shows a thin, flat, infinite sheet, carrying a uniform current specified as a surface current density $\boldsymbol{J}_s(\boldsymbol{r}, t) = J_{so}\hat{y}$ along the y-axis.

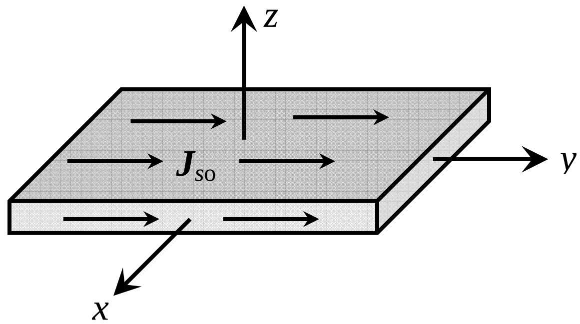

Figure 3(a)

2 pts a) Use Ampere's law, $\nabla \times \boldsymbol{H} = \boldsymbol{J}$, along with symmetry arguments to determine the strength of the magnetic field $\boldsymbol{H}(\boldsymbol{r})$ throughout the entire space.

2 pts b) Using symmetry and the fact that $\nabla \times \boldsymbol{A} = \mu_o \boldsymbol{H}$, find the vector potential $\boldsymbol{A}(\boldsymbol{r})$ throughout the entire space. (Ignore the constants of integration.)

Next, assume two identical current-carrying sheets are placed parallel to each other at a distance d_o, as shown in Fig. 3(b). The surface current densities in the two sheets are equal but opposite to each other. You may assume that the area $\mathcal{a}$ of each sheet is large enough for the edge effects to be negligible. The currents in the sheets are supplied by independent batteries, having voltages $V_1(t)$ and $V_2(t)$.

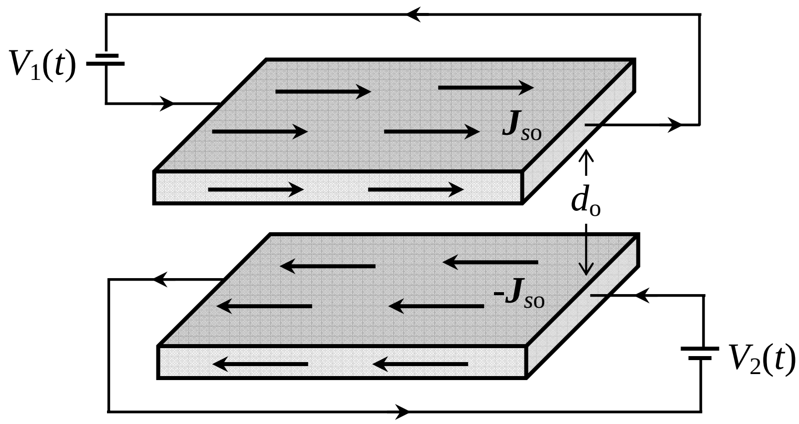

Figure 3(b)

2 pts c) Determine the magnetic field $\boldsymbol{H}(\boldsymbol{r})$ throughout the entire space.

2 pts d) The upper plate is **slowly** moved up, so that the separation between the plates is increased from d_o to d_1. The batteries are programmed to change their voltages during this operation in such a way as to maintain the current densities ($\pm J_{so}\hat{y}$) constant at all times. Verify the conservation of energy for this system by accounting for the energy stored in the magnetic field, the work performed by the moving plate on the outside world, and the energies supplied by the two batteries during the movement of the upper plate.

Hint: For the upper plate, the required additional voltage may be obtained from the Lorentz law, $\boldsymbol{F}(t) = q\,\boldsymbol{v}(t) \times \boldsymbol{B}$, which gives the force experienced by the upwardly mobile electrons; here $\boldsymbol{v}(t)$ is the instantaneous velocity of the upper plate along the vertical axis. The voltage induced in the lower plate, however, is produced by $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$.
