Opti 501 **Midterm Exam** (10/4/2005) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple the pages together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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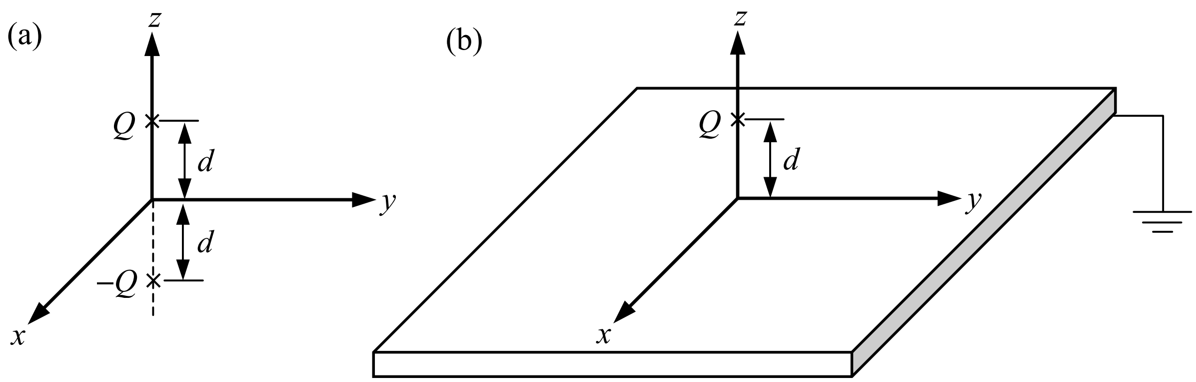
1) A pair of electric charges $\pm Q$ is placed at $(x, y, z) = (0, 0, \pm d)$, as shown in Fig. 1(a).

(2 pts) a) What is the E-field at an arbitrary point in the xy-plane, say, at $\boldsymbol{r} = (x, y, 0)$?

(2 pts) b) What is the electric potential $\psi(x, y, z = 0)$ at an arbitrary point in the xy-plane?

(2 pts) c) Assume the charge $-Q$ is replaced by a perfectly conducting plate whose top surface is in the xy-plane, as in Fig. 1(b). The plate is grounded, i.e., its potential ψ is zero. Determine the surface charge density distribution $\sigma(x, y)$ at the upper surface of the conducting plate.



Hint: In the half-space $z \geq 0$, the E-field distribution in Fig. 1(a) is the same as that in Fig. 1(b), because $\nabla^2\psi(x, y, z) = -\rho(x, y, z)/\varepsilon_o$, and also the boundary condition, $\psi(x, y, z = 0) = 0$, is the same in both cases.

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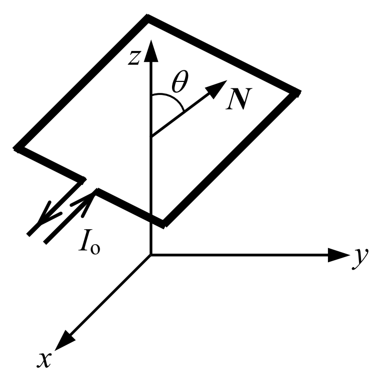
2) A square loop of conducting wire carries a constant electric current I_o. Each side of the square has length a, and its surface normal $\boldsymbol{N}$ makes an angle θ with the z-axis in the yz-plane, as shown. A uniform magnetic field $\boldsymbol{B}(x, y, z) = B_o\hat{\boldsymbol{z}}$ crosses the loop.

(1 pt) a) What is the magnetic dipole moment $\boldsymbol{m}$ of the loop (both magnitude and direction)?

(2 pts) b) Use the Lorentz law to determine the force exerted by the B-field on each side of the loop.

(2 pts) c) What is the net force and the net torque experienced by the loop?

(2 pts) d) Suppose now that the field is no longer uniform, but its magnitude varies as a function of y; in other words, B_o must now be written as $B_o(y)$. Assuming that the loop is small and that the first-order term in the Taylor series expansion of $B_o(y)$ suffices to represent the function accurately, express the net force $\boldsymbol{F}$ exerted on the loop as function of a, θ, I_o, and $\mathrm{d}B_o/\mathrm{d}y$.



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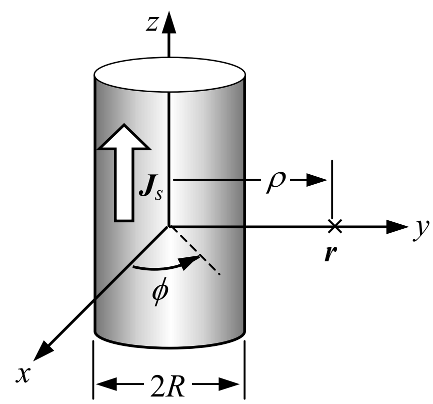
3) A thin, hollow cylinder of radius R and infinite length, centered on the z-axis, carries a constant current I_o parallel to the z-axis, as shown. The current is uniformly distributed around the cylinder's perimeter.

(1 pt) a) In terms of I_o and R, what is the surface current density $\boldsymbol{J}_s$ on the cylinder surface?

(2 pts) b) Consider the cylinder as a collection of long, thin wires, all running parallel to the z-axis. Working in cylindrical coordinates, integrate the vector potential of the wire to find the total vector potential $\boldsymbol{A}(\boldsymbol{r})$ of the cylinder at a radial distance ρ from the z-axis. (**Note**: ρ could be less than or greater than R, that is, your final answer must be valid both inside and outside the cylinder.)

(2 pts) c) Using the vector potential $\boldsymbol{A}(\boldsymbol{r})$ obtained in (b), determine the magnetic field $\boldsymbol{H}(\boldsymbol{r})$ both inside and outside the cylinder. Verify that $\boldsymbol{H}(\boldsymbol{r})$ satisfies Ampere's law as well as the boundary condition on the cylinder's surface.



Hint: $$\int_0^{\pi} \ln(1 - 2a\cos\phi + a^2)\,\mathrm{d}\phi = \begin{cases} 0 & a^2 \le 1 \\ \pi \ln(a^2) & a^2 > 1 \end{cases}$$ [Gradshteyn & Ryzhik, page 527, 4.224 (14)]

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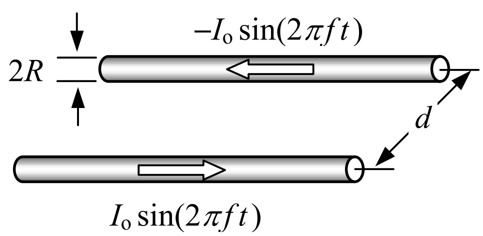
4) Two identical wires of radius R and infinite length are placed parallel to each other with a center-to-center spacing d. The wires carry the current $I(t) = I_o \sin(2\pi f t)$ in opposite directions. (The frequency f is sufficiently small that higher-order dynamic effects may be ignored.)



(2 pts) a) Find the magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ in the region of space between the two wires.

(2 pts) b) Determine the magnetic flux that crosses a rectangular area of unit length and width $(d - 2R)$ between the wires.

(1 pt) c) What is the inductance per unit length, L, of the two-wire system?

(2 pts) d) Use Faraday's law to determine the induced electric field $E(t)$ along the length of each wire.
