Opti 501 **Final Exam** (12/17/2009) **Time: 2 hours**

Please write your name and ID number on all pages, then staple them together.
Answer all questions.

Note: Bold symbols represent vectors and vector fields.

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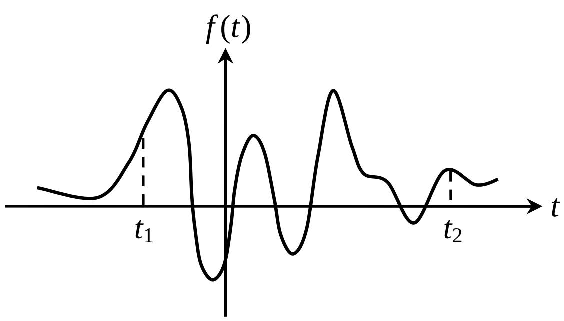
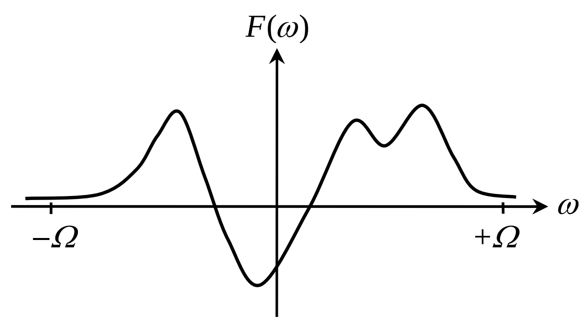
1) An arbitrary complex function $f(t)$ is defined on the time-axis t. Define the function $F(\omega)$ in the interval $-\Omega < \omega < +\Omega$ as follows:

$$F(\omega) = \int_{t_1}^{t_2} f(t') \exp(-\mathrm{i}\omega t')\, \mathrm{d}t'. \tag{1}$$

Assuming that $F(\omega)$ is well-behaved over the interval $-\Omega < \omega < +\Omega$, construct the function $\hat{f}(t)$ in accordance with the formula

$$\hat{f}(t) = (2\pi)^{-1} \int_{-\Omega}^{\Omega} F(\omega) \exp(+\mathrm{i}\omega t)\, \mathrm{d}\omega. \tag{2}$$





(4 pts) a) Substitute for $F(\omega)$ from Eq. (1) into Eq. (2), then simplify until you obtain an integral equation that relates $\hat{f}(t)$ to $f(t)$ and $\operatorname{sinc}(t)$. [**Note**: $\operatorname{sinc}(t) = \sin(\pi t)/(\pi t)$.]

(4 pts) b) Show that in the limit when $\Omega \to \infty$, assuming $f(t)$ is sufficiently well-behaved, the function $\hat{f}(t)$ approaches $f(t)$ throughout the interval $t_1 < t < t_2$.

(2 pts) c) In the limit $\Omega \to \infty$, find the values of $\hat{f}(t)$ for $t < t_1$ and $t > t_2$.

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2) Consider the flat interface between two isotropic, homogeneous, linear, semi-infinite media specified by their dielectric permittivities $(\varepsilon_a, \varepsilon_b)$ and magnetic permeabilities (μ_a, μ_b). Assume that both media are transparent at the frequency of interest ω, namely, their ε and μ are real-valued, either with ε, μ both >0 or ε, μ both <0. A homogeneous plane-wave propagating in the (ε_a, μ_a) medium, arrives at the interface with the (ε_b, μ_b) medium at an angle of incidence θ.

(5 pts) a) Let the two media have equal refractive indices but differing impedances, that is, $n_a = n_b$ but $\sqrt{\mu_a/\varepsilon_a} \neq \sqrt{\mu_b/\varepsilon_b}$. Find the Fresnel reflection coefficients ρ_p and ρ_s for p- and s-polarized light at the arbitrary angle of incidence θ. Verify that $\rho_p = \rho_s$ at normal incidence.

(5 pts) b) Let the two media now have differing refractive indices but equal impedances, that is, $n_a \neq n_b$ but $\sqrt{\mu_a/\varepsilon_a} = \sqrt{\mu_b/\varepsilon_b}$. At what angle (or angles) of incidence and for which polarization states will the reflectivity be exactly zero? Explain.

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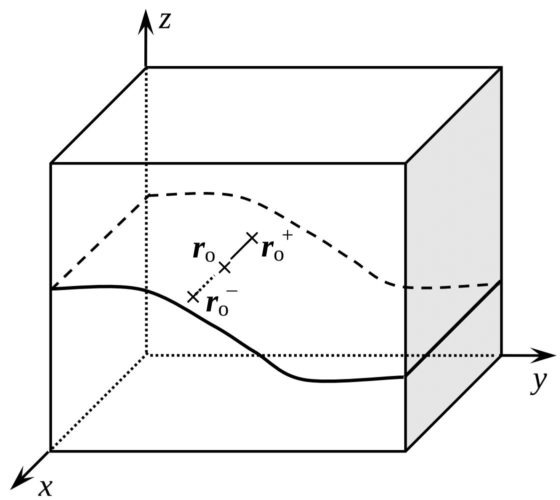
3) A region of space contains arbitrary charge, current, polarization, and magnetization distributions $\rho_{\text{free}}(\boldsymbol{r},t)$, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$, $\boldsymbol{P}(\boldsymbol{r},t)$ and $\boldsymbol{M}(\boldsymbol{r},t)$. Consider an arbitrary surface, not necessarily flat but reasonably smooth, that cuts this medium into two parts, as shown. For a given point $\boldsymbol{r}_o$ on this surface, let $\boldsymbol{r}_o^+$ be a point slightly to one side, and $\boldsymbol{r}_o^-$ a point slightly to the opposite side of the surface, with the short line segment joining $\boldsymbol{r}_o^-$ to $\boldsymbol{r}_o^+$ being perpendicular to the surface at $\boldsymbol{r}_o$. Also let the symbols $\|$ and $\perp$ indicate field components that are, respectively, parallel and perpendicular to the surface in the vicinity of $\boldsymbol{r}_o$. Each of Maxwell's equations says something about the continuity or discontinuity of the various field components, e.g., the difference between $\boldsymbol{E}_\|(\boldsymbol{r}_o^-,t)$ and $\boldsymbol{E}_\|(\boldsymbol{r}_o^+,t)$, on the two sides of the surface. These are generally known as electromagnetic boundary conditions.



(2 pts) a) Comparing D-fields at $\boldsymbol{r}_o^-$ and $\boldsymbol{r}_o^+$, what does Maxwell's 1st equation say about the continuity or discontinuity of the D-field? Explain.

(3 pts) b) Similarly, what does Maxwell's 2nd equation say about the continuity/discontinuity of the H-field across the surface? Explain.

(3 pts) c) What does Maxwell's 3rd equation say about continuity/discontinuity of the E-field? Explain.

(2 pts) d) What does Maxwell's 4th equation say about continuity/discontinuity of the B-field? Explain.

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4) Two monochromatic, homogeneous plane-waves propagate along the x-axis in an isotropic, homogeneous, linear, transparent medium specified by its relative permeability and permittivity, $\mu(\omega)$ and $\varepsilon(\omega)$, respectively. The two plane-waves have frequencies ω_1 and ω_2, with center frequency $\omega_c = \frac{1}{2}(\omega_1+\omega_2)$ and frequency difference $\Delta\omega = \omega_2 - \omega_1 \ll \omega_c$. Both plane-waves are linearly polarized along the y-axis, having amplitudes $E_1\hat{\boldsymbol{y}}\cos(k_1x-\omega_1 t)$ and $E_2\hat{\boldsymbol{y}}\cos(k_2x-\omega_2 t)$. The duration T of the "beat" may be taken as half the period of the envelope of the superposition waveform, that is, $T=\pi/\Delta\omega$.

(2 pts) a) Express k_1 and k_2 in terms of ω_1, ω_2, μ and ε.

(3 pts) b) Using the expression $\partial\mathcal{E}_E(\boldsymbol{r},t)/\partial t = \boldsymbol{E}(\boldsymbol{r},t)\cdot\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t$, find the electric energy density $\mathcal{E}_E(\boldsymbol{r},t)$ as a function of x and t. (The constant of integration may be denoted by C_o, an unknown but inconsequential parameter.)

(3 pts) c) Calculate the time-average of the energy density obtained in part (b) over the duration of a single beat, that is, from $t-(\pi/\Delta\omega)$ to t.

[Your final answer must be a function of x and t. To simplify the above calculation, you may assume that $\omega_1 = m\,\Delta\omega$ and $\omega_2 = (m+1)\Delta\omega$, where m is a large, positive integer.]

(2 pts) d) At what velocity does the time-averaged electric energy density obtained in part (c) propagate along the x-axis?

Hint: $\sin a\cos a = \frac{1}{2}\sin(2a)$; $\sin a\cos b = \frac{1}{2}\sin(a+b)+\frac{1}{2}\sin(a-b)$; $\sin a + \sin b = 2\sin[\frac{1}{2}(a+b)]\cos[\frac{1}{2}(a-b)]$.
