Opti 501 **Final Exam** (12/16/2008) **Time: 2 hours**

Please write your name and ID number on all pages, then staple them together.
Answer all questions.

Note: Bold symbols represent vectors and vector fields.

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1) A monochromatic, finite-diameter light pulse has frequency ω, cross-sectional area A, and duration τ. The pulse is long and wide, meaning that its duration is much greater than the oscillation period, i.e., $\omega\tau >> 1$, and its smallest cross-sectional diameter is much greater than a wavelength, λ. Consequently, within the space-time region where the pulse resides, it can be approximated as a plane-wave. The pulse propagates in **vacuum** along the z-axis, and its E-field amplitude is $\boldsymbol{E}_{\mathrm{o}} = E_{x\mathrm{o}}\hat{x} + E_{y\mathrm{o}}\hat{y}$.

(3 pts) a) Find the E-field and H-field energies as well as the total energy content of the pulse.

(2 pts) b) Find the time-averaged Poynting vector, $<\boldsymbol{S}(\boldsymbol{r},t)>$, and confirm that the total energy obtained in part (a) may also be derived from a knowledge of the Poynting vector.

Let the pulse arrive at **normal incidence** at the flat and smooth surface of a thick, heavy, solid block of absorptive material whose complex refractive index is $n(\omega) = 2.0 + 0.25\mathrm{i}$.

(2 pts) c) What is the total optical energy absorbed by the material?

(3 pts) d) What is the mechanical momentum acquired by the material medium after the fragment of the pulse that has entered the medium is fully absorbed?

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2) The Lorentz oscillator model of a passive, isotropic, homogeneous medium, when augmented by the Clausius-Mosotti's local field correction, yields the dielectric susceptibility $\chi(\omega)$ in the vicinity of a resonance frequency ω_{o} as follows:

$$\chi(\omega) = \frac{3\,C(\omega)}{3 - C(\omega)},$$

where

$$C(\omega) = \frac{\omega_p^2}{\omega_{\mathrm{o}}^2 - \omega^2 - \mathrm{i}\gamma\omega}; \qquad \omega_p > 0,\ \omega_{\mathrm{o}} > 0,\ \gamma \ge 0.$$

(3 pts) a) Show that the imaginary part of $C(\omega)$ is either positive or zero, but never negative, whereas its real part may assume positive, zero, or negative values depending on the physical circumstances.

(3 pts) b) Show that $\chi(\omega)$ obtained from the Clausius-Mosotti relation is similar to $C(\omega)$ in that its real part can be positive, zero, or negative, whereas its imaginary part must always be non-negative.

(3 pts) c) The complex propagation vector $\boldsymbol{\sigma}$ of a plane-wave within a passive, isotropic, homogeneous, non-magnetic medium obeys the constraint $\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = \varepsilon(\omega) = 1 + \chi(\omega)$. Assuming σ_x and σ_y are real-valued, one finds $\sigma_z = \pm\sqrt{1 + \chi(\omega) - \sigma_x^2 - \sigma_y^2}$. In light of the result obtained in part (b) above, which of the two values ($\pm$) obtained for σ_z are acceptable? Explain.

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3) Inside a homogeneous, isotropic, non-magnetic, dielectric medium of refractive index $n(\omega)$, a monochromatic, **homogeneous** plane-wave propagates along the z-axis. The plane-wave is **linearly-polarized** along the x-axis, and the medium is transparent, that is, $n(\omega)$ is real and positive.

(3 pts) a) Write expressions for the plane-wave's electric and magnetic fields, $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$, in terms of the E-field amplitude E_o, the angular frequency ω, the refractive index $n(\omega)$, the speed of light in vacuum c, and the impedance of the free space Z_o.

(2 pts) b) Express the dielectric function $\varepsilon(\omega)$ and the electric susceptibility $\chi(\omega)$ as functions of the refractive index $n(\omega)$.

(3 pts) c) Write an expression for the polarization density $\boldsymbol{P}(\boldsymbol{r},t)$ in terms of E_o, ω, c, ε_o and $n(\omega)$. What are the distributions of bound charge and current densities, $\rho_{\text{bound}}(\boldsymbol{r},t)$ and $\boldsymbol{J}_{\text{bound}}(\boldsymbol{r},t)$, in the medium?

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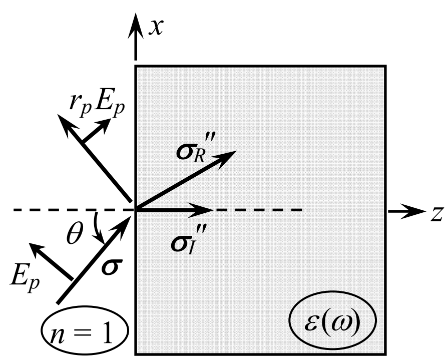
4) A p-polarized plane-wave is reflected from the flat interface between the free space and a homogeneous, isotopic, non-magnetic medium having the dielectric function $\varepsilon(\omega)$. The Fresnel reflection coefficient at the incidence angle θ is given by

$$r_p = |r_p|\exp(\mathrm{i}\phi_p) = \frac{E'_x}{E_x} = \frac{\sqrt{\varepsilon(\omega)-\sin^2\theta} - \varepsilon(\omega)\cos\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta} + \varepsilon(\omega)\cos\theta}.$$



(2 pts) a) In terms of E_p, r_p, θ, ω and c, write an expression for the z-component of the E-field, $E_z(x, z=0^-, t)$, immediately before the entrance facet of the medium.

(2 pts) b) Using the continuity of $\boldsymbol{D}_\perp$ at the interface, write an expression for $E_z(x, z=0^+, t)$ immediately beneath the entrance facet.

(2 pts) c) What is the bound surface charge density $\sigma_s(x, z=0, t)$ at the entrance facet of the medium?

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5) The refractive index of a transparent dielectric medium is given by $n(\omega) = \sqrt{1-[\omega_p^2/(\omega^2-\omega_o^2)]}$ both below and above a resonance frequency ω_o. The immediate vicinity of ω_o is excluded from consideration in order to ensure that γ, the damping coefficient in the Lorentz oscillator model, may be safely ignored. Moreover, the frequencies between ω_o and $\sqrt{\omega_o^2+\omega_p^2}$ are also excluded to ensure that $n(\omega)$ is real-valued.

(3 pts) a) Find an expression for the group refractive index $n_g(\omega) = n(\omega) + \omega[\mathrm{d}n(\omega)/\mathrm{d}\omega]$ in terms of the parameters ω_p and ω_o.

(2 pts) b) Show that, within the allowed range of frequencies, the product of $n_g(\omega)$ and $n(\omega)$ is always greater than unity.

(1 pt) c) In the frequency range $\omega > \sqrt{\omega_o^2+\omega_p^2}$, where $0 < n(\omega) < 1$, show that $n_g(\omega) > 1$.

(1 pt) d) In the frequency range $0 < \omega < \omega_o$, where $n(\omega) > 1$, show that $n_g(\omega) > 1$.
