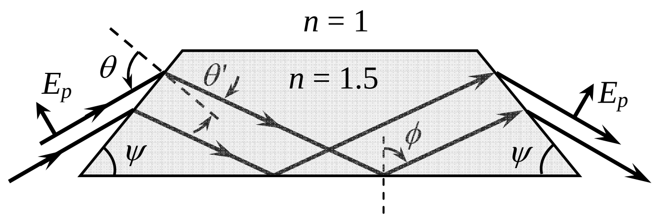
**Opti 501** **Final Exam** (12/13/2007) **Time: 2 hours**

**Please write your name and ID number on all pages, then staple them together.**
**Answer all questions.**

**Note: Bold symbols represent vectors and vector fields.**

1) A finite-diameter collimated beam arrives at an angle $\theta$ at the entrance facet of a glass prism ($n = 1.5$), as shown. The beam enters the prism, bounces off the bottom facet, then exits on the right-hand side. (The beam diameter is large enough that the effects of diffraction may be ignored; the beam may thus be treated as a plane-wave along its entire path.)

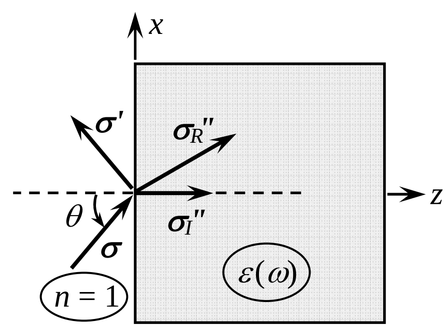



(4 pts) a) Assuming the incident beam is p-polarized, find the angle of incidence $\theta$ and a range of values for the prism angle $\psi$, such that the entire beam would emerge from the exit facet with no reflection losses whatsoever.

(3 pts) b) Using the same values for the angles $\theta$, $\phi$, $\psi$ as in part (a), what fraction of the incident optical power will be lost to reflections if the incident beam is s-polarized? (Ignore the effects of multiple reflections inside the prism.)

(2 pts) c) Answer the same question as in part (b) for a circularly-polarized incident beam.

2) A monochromatic plane wave (frequency = $\omega$) is incident from the free space at an angle $\theta$ on the flat surface of a homogeneous, isotropic, semi-infinite medium of dielectric constant $\varepsilon(\omega) = \varepsilon_R(\omega) + \mathrm{i}\,\varepsilon_I(\omega)$.



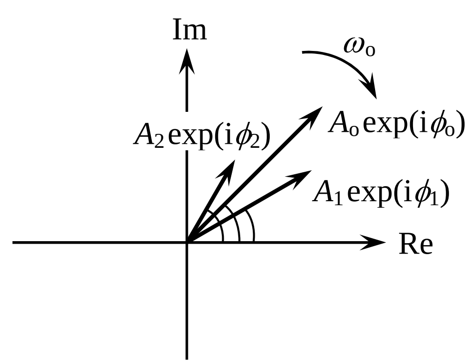

(2 pts) a) Find the propagation vector $\boldsymbol{\sigma}'' = \boldsymbol{\sigma}_R'' + \mathrm{i}\boldsymbol{\sigma}_I''$ inside the medium.

(2 pts) b) Show that at normal incidence (i.e., $\theta = 0°$), the real and imaginary components of $\boldsymbol{\sigma}''$ are parallel to each other.

(2 pts) c) At oblique incidence (i.e., $\theta \neq 0°$), show that $\boldsymbol{\sigma}_R''$ and $\boldsymbol{\sigma}_I''$ can, in general, have an arbitrary angle with each other.

(2 pts) d) Assuming $\varepsilon_I(\omega) = 0$, under what circumstances will $\boldsymbol{\sigma}_R''$ and $\boldsymbol{\sigma}_I''$ be orthogonal to each other?

3) The real-valued function $f(t)$ is obtained from the superposition of three complex functions as follows: $f(t) = \mathrm{Real}\{A_o\exp[\mathrm{i}(\phi_o - \omega_o t)] + A_1\exp[\mathrm{i}(\phi_1 - \omega_1 t)] + A_2\exp[\mathrm{i}(\phi_2 - \omega_2 t)]\}$. Although the amplitudes $A_o$, $A_1$, $A_2$ have no specific relationship with each other, the initial phase angles are such that $\phi_o$ is halfway between $\phi_1$ and $\phi_2$, that is, $\phi_1 = \phi_o - \Delta\phi$ and $\phi_2 = \phi_o + \Delta\phi$. Similarly, the frequencies are chosen such that $\omega_o$ is halfway between $\omega_1$ and $\omega_2$, namely, $\omega_1 = \omega_o - \Delta\omega$ while $\omega_2 = \omega_o + \Delta\omega$. In what follows it will be assumed that $\Delta\omega << \omega_o$. Factoring out the common frequency term, $\exp(-\mathrm{i}\omega_o t)$, one can write $f(t) = \mathrm{Real}\{g(t)\exp(-\mathrm{i}\omega_o t)\}$, where $g(t)$ is a slowly varying function of time. The oscillations of $f(t)$ are thus seen to arise from the rapid (clockwise) rotation of



$\exp(-\mathrm{i}\,\omega_o t)$ in the complex plane, while the slowly-changing $g(t)$ determines the envelope and phase of $f(t)$.

(5 pts) In terms of $\Delta\phi$ and $\Delta\omega$ find the time $t_o$ at which the envelope of $f(t)$ reaches its peak value. (**Hint**: $t_o$ can be derived from simple geometric arguments, without complicated calculations.)

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4) A light pulse propagating along the $z$-axis in a medium of refractive index $n(\omega) = n_R(\omega) + \mathrm{i}\,n_I(\omega)$ is a superposition of plane-waves in a narrow range of frequencies $(\omega_1, \omega_2)$ centered at $\omega_o$. Each plane-wave has propagation vector $\boldsymbol{\sigma}(\omega) = (\sigma_x, \sigma_y, \sigma_z) = (0, 0, n(\omega))$. For linearly-polarized light with $E$-field along the $x$-axis, the pulse's $E$-field amplitude at a given point $\boldsymbol{r}_o = (x_o, y_o, z_o)$ may be written as follows:

$$E_x(\boldsymbol{r}_o, t) = \mathrm{Real}\left\{\int_{\omega_1}^{\omega_2} E_o(\omega)\exp\{\mathrm{i}[\omega n(\omega)(z_o/c) - \omega t]\}\,\mathrm{d}\omega\right\}.$$

(3 pts) a) Separate the term containing $n_I(\omega)$ from the exponential function, treating it as a coefficient of the spectral amplitude $E_o(\omega)$. (In this problem $E_o$ is assumed to be a real-valued function of $\omega$). The remaining phase term in the exponent, namely, $[\omega n_R(\omega)(z_o/c) - \omega t]$, may be expanded in a Taylor series around the central frequency $\omega_o$. Assuming the bandwidth $(\omega_2 - \omega_1)$ is sufficiently narrow that the first two terms in the Taylor series expansion of the phase-factor suffice, write an approximate expression for $E_x(\boldsymbol{r}_o, t)$ using this Taylor series expansion. (**Note**: The first term of the series, being independent of $\omega$, may be taken outside the integral.)

(3 pts) b) Considering that the time-dependence factor $\exp[-\mathrm{i}(\omega - \omega_o)t]$ under the integral sign varies slowly compared to $\exp(-\mathrm{i}\omega_o t)$, which is outside the integral, find the time $t_o$ at which the peak of the pulse arrives at $z = z_o$. (**Hint**: At the pulse's peak, the terms under the integral should all be in-phase.)

(3 pts) c) By definition, the group velocity $V_g$ is the velocity of the peak position of the pulse. Use the result of part (b) to find an expression for $V_g$. Explain the different roles played by $n_R(\omega)$ and $n_I(\omega)$ in the pulse propagation process.

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5) The electric susceptibility of a gain medium in the vicinity of a resonance frequency $\omega_o$ can be shown (from quantum calculations) to be similar to that of a lossy medium in the Lorentz oscillator model except for the sign of the oscillator strength, which is negative for the gain medium. Thus

$$\chi_e(\omega) = \frac{\omega_p^2}{\omega^2 - \omega_o^2 + \mathrm{i}\gamma\omega}; \qquad \omega_p > 0 \text{ and } \omega_o \gg \gamma > 0.$$

Assuming that at resonance $|\chi_e(\omega_o)| = \omega_p^2/(\gamma\omega_o)$ is sufficiently small, the approximation $n(\omega) = \sqrt{1 + \chi_e(\omega)} \approx 1 + \tfrac{1}{2}\chi_e(\omega)$ will be applicable in the vicinity of $\omega_o$.

(3 pts) a) Find the real and imaginary parts of the refractive index $n(\omega)$.

(3 pts) b) Plot $n_R(\omega)$ and $n_I(\omega)$ in the vicinity of $\omega_o$. (A rough sketch will suffice.)

(3 pts) c) Determine the group velocity $V_g$ for a pulse of light centered at $\omega = \omega_o$. For the specific values of $\omega_o = 10^{15}$ rad/sec, $\omega_p = 10^{12}$ rad/sec, and $\gamma = 10^{10}$ rad/sec, find the numerical value of $V_g$.

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