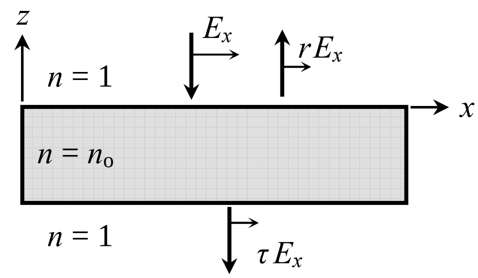
Opti 501 **Final Exam** (12/12/2006) **Time: 2 hours**

Please write your name and ID number on all pages, then staple them together.
Answer all questions.

Note: Bold symbols represent vectors and vector fields.



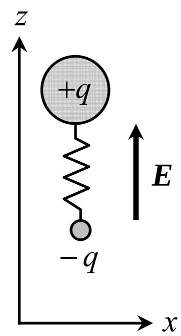
1) A monochromatic plane-wave is normally incident upon a transparent dielectric slab (i.e., real-valued refractive index n_o). The incident beam is linearly polarized, with E-field along the x-axis, as shown. The slab's reflection and transmission coefficients are r and τ, respectively.

(3 pts) a) Express the average rate of flow of optical energy $<S_z>$ (i.e., energy per unit area per unit time) in the incident beam in terms of E_x.

(3 pts) b) Show that the fraction of reflected optical energy is $R = |r|^2$, while the fraction of transmitted optical energy is $T = |\tau|^2$.

(3 pts) c) Use the conservation of energy to derive a relationship between R and T.

(3 pts) d) Use the conservation of momentum to find the radiation pressure (i.e., time-averaged force per unit area) on the slab in terms of the incident beam's $<S_z>$ and the slab's R and T.



2) In the Lorentz oscillator model, the charge $+q$ remains stationary, while the applied electric field $\boldsymbol{E}$ displaces the charge $-q$ along the z-axis. The spring constant is α, the dynamic friction coefficient is β, and the effective mass of the moving charge is m. Since the focus of this problem is on lossless (i.e., transparent) media, we will assume $\beta = 0$, that is, $\chi(\omega)$ and $\varepsilon(\omega)$ are real-valued. The force exerted by the E-field on the mobile charge is $-q\boldsymbol{E}$; denoting the displacement by Δz, the work done by the E-field on the dipole equals the change in the dipole's internal energy: $\Delta\mathcal{E} = -qE\,\Delta z = \boldsymbol{E}\cdot\Delta\boldsymbol{p}$.

(3 pts) a) Assume the E-field *slowly* rises from an initial value of zero to a final value of $\boldsymbol{E}_o$. In a linear, isotropic dielectric, therefore, the polarization density will be $\boldsymbol{P} = \varepsilon_o\chi(0)\boldsymbol{E}$. In the steady-state when the field reaches its constant value of $\boldsymbol{E}_o$, find the energy density (i.e., energy per unit volume) stored in the dipoles.

(2 pts) b) Show that the above result is consistent with the expression for the *total* E-field energy density, $\tfrac{1}{2}\varepsilon_o\varepsilon(0)|\boldsymbol{E}_o|^2$, within a uniformly polarized dielectric material.

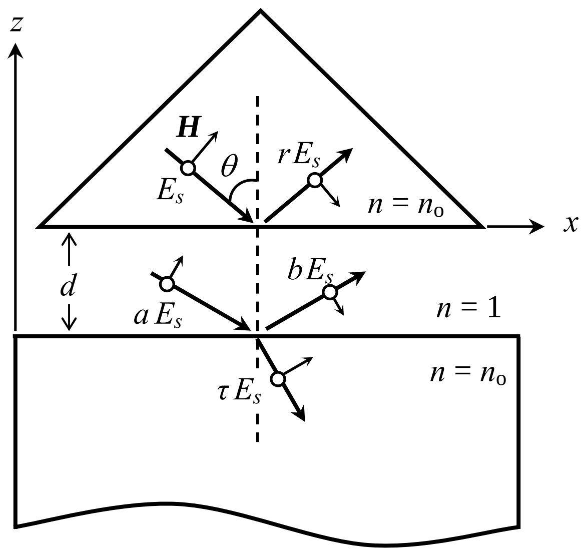
(3 pts) c) Assume now that a uniform E-field oscillates with a constant frequency ω, namely, $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_o\cos(\omega t + \phi_o)$. In this case $\boldsymbol{P} = \varepsilon_o\chi(\omega)\boldsymbol{E}$. Find the *total* E-field energy density (i.e., that of the field plus the dipoles), first as a function of time t, and then in time-averaged form.

Bonus (5 pts) d) Let $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_o[\sin(\omega_1 t) - \sin(\omega_2 t)]$ and $\boldsymbol{P}(\boldsymbol{r},t) = \varepsilon_o\boldsymbol{E}_o[\chi(\omega_1)\sin(\omega_1 t) - \chi(\omega_2)\sin(\omega_2 t)]$, where $\omega_1 = (m - \tfrac{1}{2})\Delta\omega$ and $\omega_2 = (m + \tfrac{1}{2})\Delta\omega$, with m being a large but otherwise arbitrary integer. The period of the beat signal thus produced is $T = 2\pi/\Delta\omega$. By averaging over the period T of the beat signal, determine the time-averaged *total* energy density associated with the E-field.

Hint: $2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b)$ and $2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$.

---

3) A s-polarized plane-wave is incident on the air gap between a glass prism of refractive index n_o and a glass substrate of the same index. The plane of incidence is xz, the gap-width is d, and the incidence angle θ is greater than the critical angle of total internal reflection. (Any transmission through the gap will, therefore, be a manifestation of "frustrated" total internal reflection.)



(2 pts) a) For the five plane-waves shown in the figure, express the propagation vectors $\boldsymbol{\sigma}$ in terms of θ and the refractive index n_o.

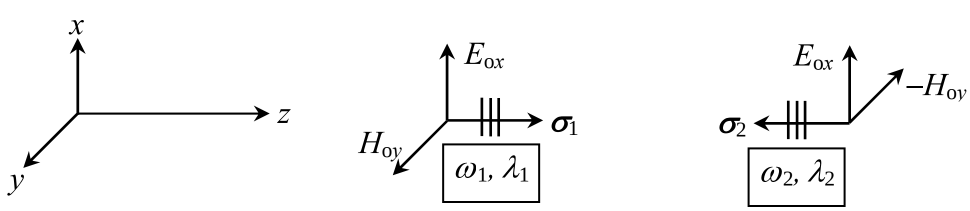
(2 pts) b) Find the magnetic field components H_x, H_z for each of the five plane-waves in terms of the corresponding E-fields, the incidence angle θ, and the refractive index n_o.

(3 pts) c) Write the continuity equations for tangential E- and H-fields at the two boundaries, namely, at $z = 0$ and $z = -d$.

(3 pts) d) Determine the reflection coefficient r by solving the four equations obtained in part (c). **Hint**: The four unknowns in these equations are r, a, b, τ.

---

4) Two linearly-polarized plane-waves, one having frequency ω_1 and free-space wavelength λ_1, the other having frequency ω_2 and free-space wavelength λ_2, propagate in opposite directions along the $\pm z$-axis. The propagation medium is free-space, and the field amplitudes of the two beams are identical (except, of course, for the H_{oy} directions, which are opposite each other). The beams interfere and set up fringes parallel to the xy-plane. By definition, $\omega_o = ½(\omega_1 + \omega_2)$ and $\Delta\omega = (\omega_2 - \omega_1)$, where $\Delta\omega << \omega_o$.



(2 pts) a) Write the real-valued E- and H-field amplitudes for each beam as functions of z and t.

(2 pts) b) Express the total E- and H-fields for the superposition of the two beams. Combine the trigonometric functions and obtain expressions containing ω_o and $\Delta\omega$.

(3 pts) c) For the superposition, find the *time-averaged* E- and H-field energy densities as functions of z and t. (Time averaging is done with respect to rapid oscillations only.) Do the energy densities remain stationary or travel along the z-axis? If they travel, specify the direction of travel.

(3 pts) d) For the two-beam superposition, find the Poynting vector $\boldsymbol{S}$ and its time average $<\boldsymbol{S}>$ (time-averaging is done over rapid oscillations only). On the average, does the total energy flow to the right, to the left, or not at all?

Hint: $\cos(a) + \cos(b) = 2\cos[½(a+b)]\cos[½(a-b)]$
$\cos(a) - \cos(b) = -2\sin[½(a+b)]\sin[½(a-b)]$
