Opti 501 **Final Exam** (12/15/2005) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple the pages together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

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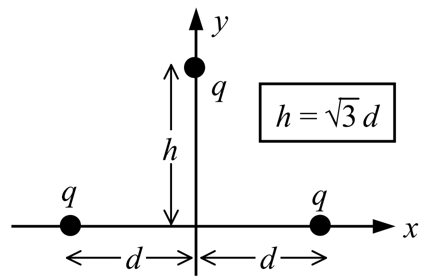
1) Three charged particles, all having the same electrical charge q, are placed (and immobilized) in the xy-plane at the vertices of an equilateral triangle, as shown.



(2 pts) ***a***) Find the coordinates of the point (x_o, y_o) at which the total electric field $\boldsymbol{E}$ is exactly equal to zero.

(2 pts) ***b***) Can (x_o, y_o) be a stable equilibrium point for a charged particle? Explain the reasoning behind your answer.

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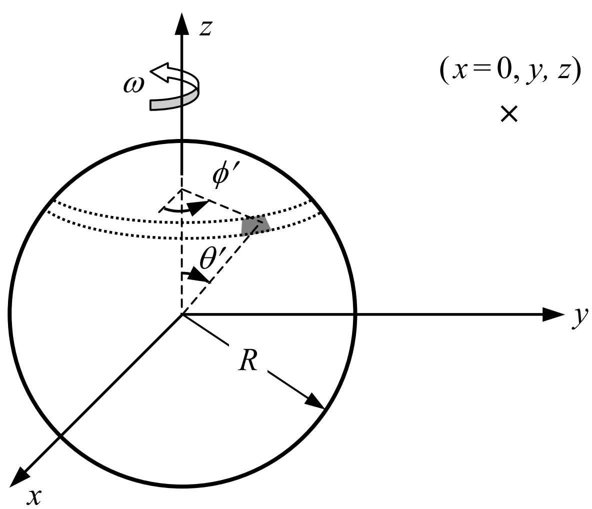
2) A hollow spherical shell of radius R and uniform surface charge density σ_o is centered at the origin of the xyz coordinate system. The shell is spinning around the z-axis at a constant angular velocity ω.

(1 pt) ***a***) What is the total charge Q of the sphere?



(2 pts) ***b***) At the point $\boldsymbol{r}' = (R, \theta', \phi')$ located on the sphere's surface, express the surface current density $\boldsymbol{J}_s(\boldsymbol{r}')$ in terms of σ_o, R, θ' and ω.

(2 pts) ***c***) Due to symmetry, the observation point $\boldsymbol{r}$ may be assumed to lie in the yz-plane, that is, $\boldsymbol{r} = (0, y, z)$. Write an expression for the distance $|\boldsymbol{r} - \boldsymbol{r}'|$ between the observation point and the point $\boldsymbol{r}'$ on the sphere surface. Assuming $|\boldsymbol{r}| >> R$, use the Taylor series expansion to approximate the *inverse* of the distance between $\boldsymbol{r}$ and $\boldsymbol{r}'$, namely, $1/|\boldsymbol{r} - \boldsymbol{r}'|$, in terms of y, z, R, θ' and ϕ'.

(2 pts) ***d***) Write an expression for the vector potential $\boldsymbol{A}$ at the observation point $\boldsymbol{r}$.
Hint: Due to symmetry, $\boldsymbol{A}(\boldsymbol{r})$ has an azimuthal component along the ϕ direction only.

(3 pts) ***e***) Assuming once again that $|\boldsymbol{r}| >> R$, simplify the double-integral in (d) using the approximate form of $1/|\boldsymbol{r} - \boldsymbol{r}'|$ obtained in (c). Determine $\boldsymbol{A}(\boldsymbol{r})$ by carrying out the necessary integrations.

(**Hint**: $\int_0^{\pi} \sin^3\theta \, d\theta = 4/3$; $\int_0^{2\pi} \sin^2\phi \, d\phi = \pi$; $\int_0^{2\pi} \sin\phi \, d\phi = 0$)

(2 pts) ***f***) Show that the spinning charged sphere behaves as a magnetic dipole when observed from sufficiently far away. Express the magnetic dipole moment $\boldsymbol{m}$ in terms of Q, R and ω.

(2 pts) ***g***) Let the spherical shell's total mass M be uniformly distributed over its surface. Show that the angular momentum of the shell around the z-axis is $\boldsymbol{L} = \frac{2}{3} M R^2 \omega \hat{\boldsymbol{z}}$.

(1 pt) ***h***) Combining the results obtained in parts (*f*) and (*g*), express the magnetic dipole moment $\boldsymbol{m}$ in terms of Q, M and $\boldsymbol{L}$.

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3) The single-oscillator Lorentz model for conduction electrons leads to the dielectric function $\varepsilon(\omega) = 1 - \omega_p^2/(\omega^2 + \mathrm{i}\gamma\omega)$, where $\omega_p = \sqrt{Nq^2/m\varepsilon_o}$ is the plasma frequency, and γ is the damping parameter. At high frequencies where $\omega >> \gamma$, the imaginary term $\mathrm{i}\gamma\omega$ is negligible (compared to ω^2), and the dielectric function may be approximated as $\varepsilon(\omega) \approx 1-(\omega_p/\omega)^2$. The material's behavior in this frequency regime is said to be *plasma-like*.

(2 pts) ***a***) Beyond the plasma frequency, where $\omega > \omega_p$ (and, of course, $\omega >> \gamma$), show that the material becomes transparent.

(1 pt) ***b***) Find the phase velocity V_p of electromagnetic waves in a transparent, plasma-like material. Show that $V_p > c$, where c is the speed of light in vacuum.

(3 pts) ***c***) Find the group velocity V_g of electromagnetic waves in a transparent plasma-like material. Show that $V_g < c$.

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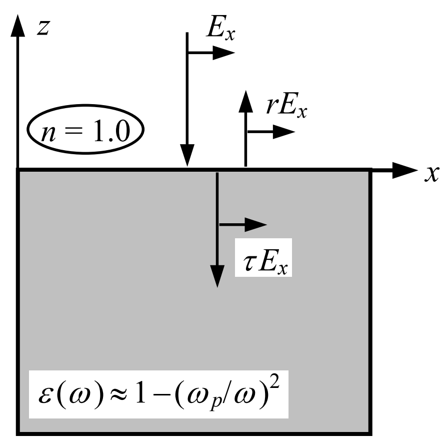
4) A homogeneous plane-wave of frequency ω, linearly polarized along the x-axis, is normally incident from the free space onto the flat surface of a *plasma-like* medium. The medium's dielectric function may be approximated as $\varepsilon(\omega) \approx 1-(\omega_p/\omega)^2$, provided that $\omega >> \gamma$. (As usual, γ is the Lorentz oscillator's damping parameter.) The concern of the present problem is the frequency regime slightly *below* the plasma frequency, where $\gamma << \omega < \omega_p$.



(3 pts) ***a***) Write the field distributions for the incident (E_x, H_y), reflected (E'_x, H'_y), and transmitted (E''_x, H''_y) beams throughout the relevant (semi-infinite) media.

(2 pts) ***b***) Match the boundary conditions at the interface between the free space and the plasma-like medium to find the reflection and transmission coefficients r and τ.

(1 pt) ***c***) Show that the reflectance $R = |r|^2$ of the plasma-like medium (in the frequency regime $\omega < \omega_p$) is 100%.

(2 pts) ***d***) Determine the time-averaged Poynting vector $<\boldsymbol{S}(\boldsymbol{r}, t)>$ in the plasma-like medium, and show that it is consistent with the 100% reflectance found in part (*c*).

(1 pt) ***e***) What is the penetration depth (or skin depth) within the plasma-like medium?

(6 pts) **5**) A homogeneous plane-wave of wavelength λ_o traveling in free space is reflected at an angle θ from the flat surface of a perfect conductor (i.e., $|n_R + \mathrm{i}\, n_I| \to \infty$). Treating the cases of p- and s-polarization separately, determine the reflection coefficient r, surface-current density $\boldsymbol{J}_s(\boldsymbol{r}, t)$, and surface-charge density $\sigma_s(\boldsymbol{r}, t)$ in each case.