Opti 501 **Final Exam** (12/15/2011) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

Note: Bold symbols represent vectors and vector fields.

Problem 1)

2 Pts a) The function $g_\alpha(x) = \alpha \exp(-\alpha|x|)$ is defined for real-valued and positive α over the entire x-axis. To what well-known function does $g_\alpha(x)$ approach in the limit when $\alpha \to \infty$?

2 Pts b) Expanding the arbitrary function $f(x)$ in a Taylor series around $x=0$, evaluate the integral $\int_{-\infty}^{\infty} f(x) g_\alpha(x)\,dx$ in the limit when $\alpha \to \infty$.

3 Pts c) The function $h_\beta(x) = \beta^{3/2} x \exp(-\beta x^2)$ is defined for real-valued and positive β over the entire x-axis. To what well-known function does $h_\beta(x)$ approach in the limit when $\beta \to \infty$?

3 Pts d) Expanding the arbitrary function $f(x)$ in a Taylor series around $x=0$, evaluate the integral $\int_{-\infty}^{\infty} f(x) h_\beta(x)\,dx$ in the limit when $\beta \to \infty$.

Hint: $\int_0^\infty x^n \exp(-\alpha x)\,dx = \frac{n!}{\alpha^{n+1}}$; Re$(\alpha) > 0$, $n = 0, 1, 2, \ldots$ (G&R 3.351–3),

$n! = 1\cdot 2\cdot 3 \cdots n$, with 0! defined as 1.

$\int_0^\infty x^{2n} \exp(-\beta x^2)\,dx = \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}\beta^{n+\frac{1}{2}}}$; $\beta > 0$, $n = 0, 1, 2, \ldots$ (G&R 3.461–2),

$(2n-1)!! = 1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)$; for $n=0$, $(2n-1)!!$ is defined as 1.

Problem 2) A spherical shell of radius R and uniform surface-charge-density σ_{so} spins around the z-axis at a constant angular velocity ω_0.

4 Pts a) Find the total charge Q and the magnetic dipole moment m_z of the spinning shell.

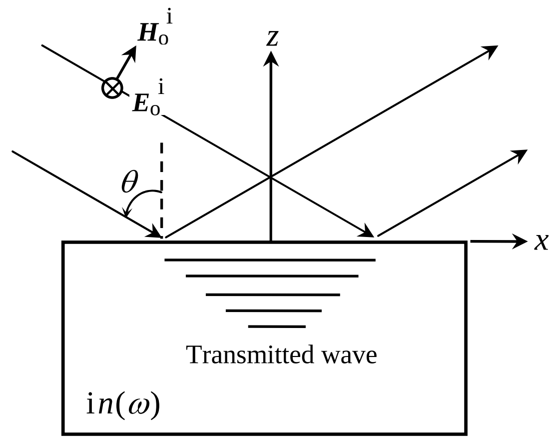
6 Pts b) The electric and magnetic fields inside and outside the shell are known to be

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_o r^2}\hat{\boldsymbol{r}}; & r > R, \\ 0; & r < R. \end{cases} \qquad \boldsymbol{B}(\boldsymbol{r}) = \begin{cases} \dfrac{m_z(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{4\pi r^3}; & r > R, \\ \dfrac{2}{3}\dfrac{m_z}{(4\pi R^3/3)}\hat{\boldsymbol{z}}; & r < R. \end{cases}$$

Determine the total energy associated with the E- and H-fields, and also the *electromagnetic* angular momentum $\mathcal{L}_z$ of the spinning shell.

Hint: $\int_0^\pi \sin^3\theta\,d\theta = \frac{4}{3}$.

Problem 3) A monochromatic plane-wave of frequency ω arrives from free space at the surface of a plasma-like medium at the oblique incidence angle θ. The incident beam is *s*-polarized, and the medium is specified by its permeability $\mu(\omega)=1.0$ and permittivity $\varepsilon(\omega)=1-(\omega_p/\omega)^2$. At frequencies below the plasma frequency ω_p, the dielectric function of the medium is negative-valued and, therefore, its purely imaginary refractive index may be written as i$n(\omega)$, where $n(\omega)$ is real and positive.



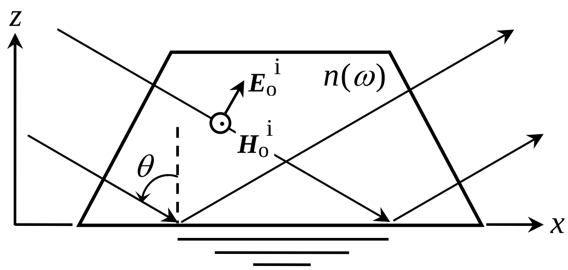
3 Pts a) Write expressions for the incident, reflected, and transmitted plane-waves, specifying the various components of the *E*- and *H*-fields in terms of the corresponding *E*-field component along the *y*-axis.

3 Pts b) Match the boundary conditions at the surface to determine the *E*- and *H*-fields of the reflected and transmitted waves in terms of θ, $n(\omega)$, and the incident *E*-field amplitude E_{yo}^{i}.

2 Pts c) Show that the reflectance of the plasma-like medium is 100% at all angles of incidence.

2 Pts d) What is the penetration-depth of the fields into the plasma-like medium? Is there energy stored in these fields? What is the *time-averaged* Poynting vector in the plasma-like medium?

Problem 4) A monochromatic plane-wave arrives at the interface between a glass prism and air at the oblique angle $\theta>\theta_c$, as shown. The incident beam is *p*-polarized, and the prism material has $\mu(\omega)=1$ and $\varepsilon(\omega)=n^2(\omega)$, with $n(\omega)>1$ being the real-valued refractive index of the prism at the frequency ω of the incident beam.



3 Pts a) Write expressions for the incident, reflected, and transmitted plane-waves, specifying the various components of the *E*- and *H*-fields in terms of the corresponding *E*-field component along the *x*-axis.

3 Pts b) Match the boundary conditions at the interface to derive the *E*- and *H*-fields of the evanescent wave in terms of θ, $n(\omega)$, and the incident *E*-field amplitude E_{xo}^{i}.

4 Pts c) Calculate the total *time-averaged* stored energy (per unit-area of the interface) in the *E*- and *H*-fields of the evanescent wave.
