Opti 501 **Midterm Exam Solutions** 4/20/2010

Problem 1) a) The standard Maxwell's equations are

$$\nabla\cdot\boldsymbol{D}(\boldsymbol{r},t)=\rho_{\text{free}}(\boldsymbol{r},t), \tag{1a}$$

$$\nabla\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t, \tag{1b}$$

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t, \tag{1c}$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r},t)=0. \tag{1d}$$

To eliminate $\boldsymbol{E}$ and $\boldsymbol{B}$, one need only modify the third and fourth equations, as follows:

Eq. (1c): $\varepsilon_o\nabla\times\boldsymbol{E}(\boldsymbol{r},t)+\nabla\times\boldsymbol{P}(\boldsymbol{r},t)=-\varepsilon_o\partial\boldsymbol{M}(\boldsymbol{r},t)/\partial t-\varepsilon_o\mu_o\partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t+\nabla\times\boldsymbol{P}(\boldsymbol{r},t)$

$$\Rightarrow\ \nabla\times\boldsymbol{D}(\boldsymbol{r},t)=-\varepsilon_o[\partial\boldsymbol{M}(\boldsymbol{r},t)/\partial t-\varepsilon_o^{-1}\nabla\times\boldsymbol{P}(\boldsymbol{r},t)]-\varepsilon_o\mu_o\partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t. \tag{1c$'$}$$

Eq. (1d): $\mu_o\nabla\cdot\boldsymbol{H}(\boldsymbol{r},t)=-\nabla\cdot\boldsymbol{M}(\boldsymbol{r},t).$ (1d′)

b) Transforming the modified equations to the Fourier domain yields,

$$\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{D}(\boldsymbol{k},\omega)=\rho_{\text{free}}(\boldsymbol{k},\omega), \tag{2a}$$

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)=\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\,\omega\boldsymbol{D}(\boldsymbol{k},\omega), \tag{2b}$$

$$\boldsymbol{k}\times\boldsymbol{D}(\boldsymbol{k},\omega)=\varepsilon_o\omega\boldsymbol{M}(\boldsymbol{k},\omega)+\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega)+(\omega/c^2)\boldsymbol{H}(\boldsymbol{k},\omega), \tag{2c}$$

$$\mu_o\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega)=-\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega). \tag{2d}$$

c) Cross-multiplying "$-\mathrm{i}\boldsymbol{k}$" into Eq. (2b), one arrives at

$$\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)]=-\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\omega\boldsymbol{k}\times\boldsymbol{D}(\boldsymbol{k},\omega). \tag{3}$$

Using the vector identity $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H})=(\boldsymbol{k}\cdot\boldsymbol{H})\boldsymbol{k}-k^2\boldsymbol{H}$ in the preceding equation, one obtains

$$[\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega)]\boldsymbol{k}-k^2\boldsymbol{H}(\boldsymbol{k},\omega)=-\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\omega\boldsymbol{k}\times\boldsymbol{D}(\boldsymbol{k},\omega). \tag{4}$$

Substitution from Eqs. (2c) and (2d) into Eq. (4) then yields

$$-\mu_o^{-1}[\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)]\boldsymbol{k}-k^2\boldsymbol{H}(\boldsymbol{k},\omega)=-\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\varepsilon_o\omega^2\boldsymbol{M}(\boldsymbol{k},\omega)-\omega\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega)-(\omega^2/c^2)\boldsymbol{H}(\boldsymbol{k},\omega)$$

$$\Rightarrow\ \boxed{\boldsymbol{H}(\boldsymbol{k},\omega)=\{\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)+\omega\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega)+\varepsilon_o\omega^2\boldsymbol{M}(\boldsymbol{k},\omega)-\mu_o^{-1}[\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)]\boldsymbol{k}\}/(k^2-\omega^2/c^2).} \tag{5}$$

To determine $\boldsymbol{D}(\boldsymbol{k},\omega)$, proceed along similar lines, namely, cross-multiply $\boldsymbol{k}$ into Eq. (2c), use the vector identity $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{D})=(\boldsymbol{k}\cdot\boldsymbol{D})\boldsymbol{k}-k^2\boldsymbol{D}$, then substitute from Eqs. (2a) and (2b) into the resulting equation to obtain

$$\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{D}(\boldsymbol{k},\omega)]=\varepsilon_o\omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)+\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega)]+(\omega/c^2)\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)$$

$$\Rightarrow\ [\boldsymbol{k}\cdot\boldsymbol{D}(\boldsymbol{k},\omega)]\boldsymbol{k}-k^2\boldsymbol{D}(\boldsymbol{k},\omega)=\varepsilon_o\omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)+[\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k}-k^2\boldsymbol{P}(\boldsymbol{k},\omega)$$
$$-\mathrm{i}(\omega/c^2)\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-(\omega^2/c^2)\boldsymbol{D}(\boldsymbol{k},\omega)$$

$$\Rightarrow\ \boxed{\begin{aligned}\boldsymbol{D}(\boldsymbol{k},\omega)=\{&-\mathrm{i}\rho_{\text{free}}(\boldsymbol{k},\omega)\boldsymbol{k}+\mathrm{i}(\omega/c^2)\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)+k^2\boldsymbol{P}(\boldsymbol{k},\omega)-[\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k}\\&-\varepsilon_o\omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)\}/(k^2-\omega^2/c^2).\end{aligned}} \tag{6}$$

Problem 2)

a) $\rho(\boldsymbol{r},t) = q\,\delta(x-Vt)\,\delta(y)\,\delta(z);\qquad \boldsymbol{J}(\boldsymbol{r},t) = qV\,\delta(x-Vt)\,\delta(y)\,\delta(z)\,\hat{\boldsymbol{x}}.$

b) $$\rho(\boldsymbol{k},\omega) = \iiiint_{-\infty}^{\infty} \rho(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t = q\iint_{-\infty}^{\infty}\delta(x-Vt)\exp[-\mathrm{i}(k_x x-\omega t)]\,\mathrm{d}x\,\mathrm{d}t$$

$$= q\int_{-\infty}^{\infty}\exp[\mathrm{i}(\omega - Vk_x)t]\,\mathrm{d}t = 2\pi q\,\delta(\omega - Vk_x).$$

$$\boldsymbol{J}(\boldsymbol{k},\omega) = \iiiint_{-\infty}^{\infty} \boldsymbol{J}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t = qV\hat{\boldsymbol{x}}\iint_{-\infty}^{\infty}\delta(x-Vt)\exp[-\mathrm{i}(k_x x-\omega t)]\,\mathrm{d}x\,\mathrm{d}t$$

$$= qV\hat{\boldsymbol{x}}\int_{-\infty}^{\infty}\exp[\mathrm{i}(\omega - Vk_x)t]\,\mathrm{d}t = 2\pi qV\,\delta(\omega - Vk_x)\,\hat{\boldsymbol{x}}.$$

The scalar and vector potentials are thus given by

$$\psi(\boldsymbol{k},\omega) = \varepsilon_o^{-1}\rho(\boldsymbol{k},\omega)/(k^2-\omega^2/c^2) = (2\pi q/\varepsilon_o)\delta(\omega - Vk_x)/(k^2-\omega^2/c^2);$$

$$\boldsymbol{A}(\boldsymbol{k},\omega) = \mu_o\boldsymbol{J}(\boldsymbol{k},\omega)/(k^2-\omega^2/c^2) = (2\pi\mu_o qV\hat{\boldsymbol{x}})\,\delta(\omega - Vk_x)/(k^2-\omega^2/c^2).$$

c) Inverse Fourier transforming the scalar potential, we find

$$\psi(\boldsymbol{r},t) = (2\pi)^{-4}\iiiint_{-\infty}^{\infty}\psi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$= (2\pi)^{-3}(q/\varepsilon_o)\iiiint_{-\infty}^{\infty}(k^2-\omega^2/c^2)^{-1}\delta(\omega - Vk_x)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$= (2\pi)^{-3}(q/\varepsilon_o)\iiint_{-\infty}^{\infty}[(1-V^2/c^2)k_x^2+k_y^2+k_z^2]^{-1}\exp\{\mathrm{i}[k_x(x-Vt)+k_y y+k_z z]\}\,\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z$$

Defining the parameter $\gamma = 1/\sqrt{1-(V/c)^2}$, then changing the variable from k_x to k_x/γ yields

$$\psi(\boldsymbol{r},t) = (2\pi)^{-3}(\gamma q/\varepsilon_o)\iiint_{-\infty}^{\infty}(k_x^2+k_y^2+k_z^2)^{-1}\exp\{\mathrm{i}[k_x\gamma(x-Vt)+k_y y+k_z z]\}\,\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z$$

$$= (2\pi)^{-3}(\gamma q/\varepsilon_o)\iiint_{-\infty}^{\infty}k^{-2}\exp\{\mathrm{i}\boldsymbol{k}\cdot[\gamma(x-Vt)\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}]\}\,\mathrm{d}\boldsymbol{k}$$

$$= (2\pi)^{-2}(\gamma q/\varepsilon_o)\int_0^{\infty}\mathrm{d}k\int_0^{\pi}\sin\theta\,\exp[\mathrm{i}k\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}\cos\theta]\,\mathrm{d}\theta$$

$$= (\gamma q/2\pi^2\varepsilon_o)\int_0^{\infty}\{\sin[k\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}]/[k\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}]\}\,\mathrm{d}k$$

$$= \gamma q/[4\pi\varepsilon_o\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}]$$

Similarly, the inverse Fourier transform of $\boldsymbol{A}(\boldsymbol{k},\omega)$ is found to be

$$\boldsymbol{A}(\boldsymbol{r},t) = \mu_o\gamma qV\hat{\boldsymbol{x}}/[4\pi\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}].$$

d) The fields are found using $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t) - \partial\boldsymbol{A}(\boldsymbol{r},t)/\partial t$ and $\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t)$, as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = (\gamma q/4\pi\varepsilon_o)[(x-Vt)\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}]/[\gamma^2(x-Vt)^2+y^2+z^2]^{3/2};$$

$$\boldsymbol{B}(\boldsymbol{r},t) = (\mu_o q\gamma V/4\pi)(-z\hat{\boldsymbol{y}}+y\hat{\boldsymbol{z}})/[\gamma^2(x-Vt)^2+y^2+z^2]^{3/2}.$$

Problem 3)

a) $\boldsymbol{P}(\boldsymbol{r},t) = p_o\delta(x)\,\delta(y)\,\delta(z)[\cos(\omega_o t)\hat{\boldsymbol{x}} + \sin(\omega_o t)\hat{\boldsymbol{y}}]$.

b) $\rho_b^{(e)}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t) = -\partial P_x/\partial x - \partial P_y/\partial y = -p_o[\delta'(x)\,\delta(y)\cos(\omega_o t) + \delta(x)\,\delta'(y)\sin(\omega_o t)]\,\delta(z)$.

$\boldsymbol{J}_b^{(e)}(\boldsymbol{r},t) = \partial\boldsymbol{P}(\boldsymbol{r},t)/\partial t = -p_o\omega_o\,\delta(x)\,\delta(y)\,\delta(z)[\sin(\omega_o t)\hat{\boldsymbol{x}} - \cos(\omega_o t)\hat{\boldsymbol{y}}]$.

c) Continuity equation:

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_b^{(e)} + \partial\rho_b^{(e)}/\partial t = -p_o\omega_o[\delta'(x)\,\delta(y)\,\delta(z)\sin(\omega_o t) - \delta(x)\,\delta'(y)\,\delta(z)\cos(\omega_o t)]$$
$$-p_o\omega_o[-\delta'(x)\,\delta(y)\sin(\omega_o t) + \delta(x)\,\delta'(y)\cos(\omega_o t)]\,\delta(z) = 0.$$

d) For the electric point-dipole $p_o\cos(\omega_o t)\hat{\boldsymbol{z}}$ aligned with the z-axis, the potentials are given in a spherical coordinate system. In the present problem, however, we have two oscillating dipoles, one aligned with the x-axis, having magnitude $p_o\cos(\omega_o t)$, the other aligned with the y-axis and having magnitude $p_o\sin(\omega_o t)$. Retaining the same spherical coordinate system in which θ is measured from the z-axis, we recognize that, for the first dipole, $\cos\theta$ in the expression of the scalar potential must be replaced with $\sin\theta\cos\phi$, while for the second dipole it must be replaced with $\sin\theta\sin\phi$. Also, for the second dipole, the origin of time t must be shifted by one quarter of one period such that $\cos(\omega_o t)$ is turned into $\sin(\omega_o t)$, in which case $\sin(\omega_o t)$ appearing in the expressions of $\psi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$ must undergo a corresponding shift to become $-\cos(\omega_o t)$. Subsequently we add the respective potentials of the two dipoles to obtain

$$\boldsymbol{A}(\boldsymbol{r},t) = -(\mu_o p_o\omega_o/4\pi r)\{\sin[\omega_o(t-r/c)]\hat{\boldsymbol{x}} - \cos[\omega_o(t-r/c)]\hat{\boldsymbol{y}}\};$$

$$\psi(\boldsymbol{r},t) = (p_o\sin\theta\cos\phi/4\pi\varepsilon_o r^2)\{\cos[\omega_o(t-r/c)] - (\omega_o r/c)\sin[\omega_o(t-r/c)]\}$$
$$+ (p_o\sin\theta\sin\phi/4\pi\varepsilon_o r^2)\{\sin[\omega_o(t-r/c)] + (\omega_o r/c)\cos[\omega_o(t-r/c)]\}$$
$$= (p_o\sin\theta/4\pi\varepsilon_o r^2)\{\cos[\omega_o(t-r/c)-\phi] - (\omega_o r/c)\sin[\omega_o(t-r/c)-\phi]\}.$$