Problem 1) $\boldsymbol{M}(\boldsymbol{r},t)=M_o\hat{\boldsymbol{z}}\,\text{Sphere}(r/R)$ is the precise representation of the magnetization distribution, which, in spherical coordinates, is written $\boldsymbol{M}(\boldsymbol{r},t)=M_o\text{Sphere}(r/R)(\cos\theta\hat{\boldsymbol{r}}-\sin\theta\hat{\boldsymbol{\theta}})$.

a) $$\rho_{\text{bound}}^{(m)}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r},t)=-\frac{1}{r^2}\frac{\partial(r^2M_r)}{\partial r}-\frac{1}{r\sin\theta}\frac{\partial(\sin\theta M_\theta)}{\partial\theta}$$

$$=-\frac{M_o\cos\theta}{r^2}[2r\,\text{Sphere}(r/R)-r^2\delta(r-R)]+\frac{M_o\,\text{Sphere}(r/R)}{r\sin\theta}(2\sin\theta\cos\theta)$$

$$=M_o\delta(r-R)\cos\theta.$$

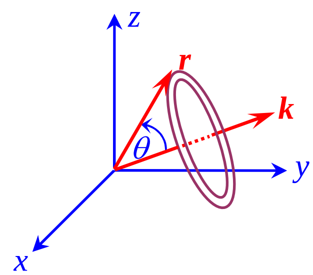
This surface-charge-density is positive on the upper hemisphere and negative on the lower hemisphere, changing continuously from maximum at the north-pole, to zero at the equator, to minimum at the south-pole.

b) $$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t)=\mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)=\frac{\mu_o^{-1}}{r}\left[\frac{\partial(r\,M_\theta)}{\partial r}-\frac{\partial M_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$=\frac{\mu_o^{-1}M_o}{r}\{-[\text{Sphere}(r/R)-r\,\delta(r-R)]\sin\theta+\text{Sphere}(r/R)\sin\theta\}\hat{\boldsymbol{\phi}}$$

$$=\mu_o^{-1}M_o\delta(r-R)\sin\theta\hat{\boldsymbol{\phi}}.$$

This azimuthal surface-current-density is zero at the north-pole, increases to a maximum at the equator, then decreases again to zero at the south-pole.



c) $$\boldsymbol{M}(\boldsymbol{k},\omega)=\int_{-\infty}^{\infty}M_o\hat{\boldsymbol{z}}\,\text{Sphere}(r/R)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\mathrm{d}\boldsymbol{r}\mathrm{d}t$$

$$=M_o\hat{\boldsymbol{z}}[2\pi\delta(\omega)]\int_{r=0}^{R}\int_{\theta=0}^{\pi}\exp(-\mathrm{i}kr\cos\theta)2\pi r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta$$

$$=4\pi^2M_o\hat{\boldsymbol{z}}\,\delta(\omega)\int_{r=0}^{R}\left.\frac{\exp(-\mathrm{i}kr\cos\theta)}{\mathrm{i}kr}\right|_{\theta=0}^{\pi}r^2\mathrm{d}r$$

$$=8\pi^2M_ok^{-1}\delta(\omega)\hat{\boldsymbol{z}}\int_{r=0}^{R}r\sin(kr)\mathrm{d}r$$

Integration by parts →
$$=8\pi^2M_ok^{-1}\delta(\omega)\hat{\boldsymbol{z}}\left[-k^{-1}r\cos(kr)\Big|_{r=0}^{R}+\int_{r=0}^{R}k^{-1}\cos(kr)\mathrm{d}r\right]$$

$$=8\pi^2M_ok^{-1}\delta(\omega)\hat{\boldsymbol{z}}\left[-k^{-1}R\cos(kR)+k^{-2}\sin(kR)\right]$$

$$=8\pi^2M_ok^{-3}[\sin(kR)-kR\cos(kR)]\delta(\omega)\hat{\boldsymbol{z}}.$$

Problem 2) a) The large-argument approximate forms of Bessel functions of the first and second kinds are $J_n(x)\approx\sqrt{2/(\pi x)}\,\cos[x-(n\pi/2)-(\pi/4)]$ and $Y_n(x)\approx\sqrt{2/(\pi x)}\,\sin[x-(n\pi/2)-(\pi/4)]$. Substitution into the expressions for the E- and H-fields then yields

$$\boldsymbol{E}(\boldsymbol{r},t)\simeq-\tfrac{1}{4}\mu_o I_o\omega_o\sqrt{2c/(\pi\rho\omega_o)}\,[\cos(\rho\omega_o/c-\pi/4)\cos(\omega_o t)+\sin(\rho\omega_o/c-\pi/4)\sin(\omega_o t)]\hat{\boldsymbol{z}}$$

$$=-\frac{Z_o I_o}{\sqrt{4\lambda_o\rho}}\cos[\omega_o(t-\rho/c)+\pi/4]\hat{\boldsymbol{z}}.$$

$$\boldsymbol{H}(\boldsymbol{r},t)\simeq\frac{I_o\omega_o}{4c}\sqrt{2c/(\pi\rho\omega_o)}\,[\cos(\rho\omega_o/c-3\pi/4)\sin(\omega_o t)-\sin(\rho\omega_o/c-3\pi/4)\cos(\omega_o t)]\hat{\boldsymbol{\phi}}$$

$$=\frac{I_o}{\sqrt{4\lambda_o\rho}}\cos[\omega_o(t-\rho/c)+\pi/4]\hat{\boldsymbol{\phi}}.$$

b) $$\boldsymbol{S}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)\simeq\frac{Z_o I_o^2}{4\lambda_o\rho}\cos^2[\omega_o(t-\rho/c)+\pi/4]\hat{\boldsymbol{\rho}}\qquad\text{(far field).}$$

c) $$\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle\simeq\frac{Z_o I_o^2}{4\lambda_o\rho}\langle\cos^2[\omega_o(t-\rho/c)+\pi/4]\rangle\hat{\boldsymbol{\rho}}=\frac{Z_o I_o^2}{8\lambda_o\rho}\hat{\boldsymbol{\rho}}\qquad\text{(far field).}$$

The time-averaged energy leaving a cylinder of radius R and height L per second is obtained by multiplying the above time-averaged Poynting vector at $\rho=R$ with the surface area $2\pi RL$ of the cylinder. The result, $\pi Z_o I_o^2 L/(4\lambda_o)$, is clearly independent of the cylinder radius, as it should be, considering that the electromagnetic power radiated by the wire must leave the surrounding cylinder, irrespective of the cylinder radius.

Problem 3) a) On the cylindrical walls of the cavity, where $\rho=R$, the tangential E-field (E_z in the present case) must vanish. Therefore, $J_0(R\omega_o/c)=0$. Acceptable values of R are thus $R_n=cx_n/\omega_o$.

b) $$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t\quad\rightarrow\quad-(\partial E_z/\partial\rho)\hat{\boldsymbol{\phi}}=-\mu_o\partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t$$

$$\rightarrow\ \partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t=\mu_o^{-1}E_o\,(\omega_o/c)\,J_0'(\rho\omega_o/c)\cos(\omega_o t)\hat{\boldsymbol{\phi}}=-(\omega_o E_o/Z_o)\,J_1(\rho\omega_o/c)\cos(\omega_o t)\hat{\boldsymbol{\phi}}$$

$$\rightarrow\ \boldsymbol{H}(\boldsymbol{r},t)=-(E_o/Z_o)\,J_1(\rho\omega_o/c)\sin(\omega_o t)\hat{\boldsymbol{\phi}}.$$

c) Maxwell's 1st equation: $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0\ \rightarrow\ \partial E_z/\partial z=0.$ Checks.

Maxwell's 2nd equation: $\boldsymbol{\nabla}\times\boldsymbol{H}=\partial\boldsymbol{D}/\partial t\ \rightarrow\ \dfrac{1}{\rho}\dfrac{\partial(\rho H_\phi)}{\partial\rho}\hat{\boldsymbol{z}}=-\varepsilon_o\omega_o E_o J_0(\rho\omega_o/c)\sin(\omega_o t)\hat{\boldsymbol{z}}$

$$\rightarrow\ -(E_o/Z_o)[(1/\rho)J_1(\rho\omega_o/c)+(\omega_o/c)J_1'(\rho\omega_o/c)]\sin(\omega_o t)\hat{\boldsymbol{z}}=-\varepsilon_o\omega_o E_o J_0(\rho\omega_o/c)\sin(\omega_o t)\hat{\boldsymbol{z}}$$

$$\rightarrow\ -(E_o/Z_o)(\omega_o/c)J_0(\rho\omega_o/c)\sin(\omega_o t)\hat{\boldsymbol{z}}=-\varepsilon_o\omega_o E_o J_0(\rho\omega_o/c)\sin(\omega_o t)\hat{\boldsymbol{z}}.\quad\text{Checks.}$$

Chapter 3, Eq. (41) $\qquad$ $1/(Z_o c)=\varepsilon_o$

Maxwell's 4th equation: $\boldsymbol{\nabla}\cdot\boldsymbol{H}=0 \quad \rightarrow \quad (1/\rho)\,\partial H_\phi/\partial\phi=0.$ Checks.

d) On the cylindrical surface of the cavity $E_r=0$; no surface-charges therefore reside on this wall, that is, $\sigma_s(\rho=R,\phi,z,t)=0.$ The tangential H-field, however, is non-zero, yielding the following surface-current-density: $\boldsymbol{J}_s(\rho=R,\phi,z,t)=-H_\phi(\rho=R,\phi,z,t)\hat{\boldsymbol{z}}=(E_o/Z_o)\,J_1(R\omega_o/c)\sin(\omega_o t)\hat{\boldsymbol{z}}.$

At the top and bottom surfaces, the perpendicular D-field is $\varepsilon_o E_z(\rho,\phi,z=\pm L/2,t)\hat{\boldsymbol{z}}$. The surface-charge-density is thus given by $\sigma_s(\rho,\phi,z=\pm L/2,t)=\mp\varepsilon_o E_o J_0(\rho\omega_o/c)\cos(\omega_o t).$ Similarly, the surface-current-density is related to the tangential component of the H-field, as follows: $\boldsymbol{J}_s(\rho,\phi,z=\pm L/2,t)=\pm H_\phi(\rho,\phi,z=\pm L/2,t)\hat{\boldsymbol{\rho}}=\mp(E_o/Z_o)\,J_1(\rho\omega_o/c)\sin(\omega_o t)\hat{\boldsymbol{\rho}}.$

e) On the cylindrical surface, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_s=0$ and $\sigma_s=0$; therefore, the continuity equation is satisfied. Also, at the top and bottom flat surfaces we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_s=\frac{1}{\rho}\frac{\partial(\rho J_{s\rho})}{\partial\rho}=\mp(E_o/Z_o)\left[(1/\rho)J_1(\rho\omega_o/c)+(\omega_o/c)J_1'(\rho\omega_o/c)\right]\sin(\omega_o t)$$

$$=\mp\varepsilon_o\omega_o E_o J_0(\rho\omega_o/c)\sin(\omega_o t),$$

$$\partial\sigma_s/\partial t=\pm\varepsilon_o\omega_o E_o J_0(\rho\omega_o/c)\sin(\omega_o t).$$

Clearly, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_s(\boldsymbol{r},t)+\partial\sigma_s(\boldsymbol{r},t)/\partial t=0$ on the flat surfaces as well.
