Opti 501 **2^{nd} Midterm Solutions** (10/28/2021) **Time: 75 minutes**

Problem 1) a) $\mathcal{F}\{\text{Rect}(t)\} = \int_{-\infty}^{\infty} \text{Rect}(t)e^{\mathrm{i}\omega t}\mathrm{d}t = \int_{-½}^{½} e^{\mathrm{i}\omega t}\mathrm{d}t = (\mathrm{i}\omega)^{-1}e^{\mathrm{i}\omega t}\Big|_{t=-½}^{½} = \dfrac{e^{½\mathrm{i}\omega} - e^{-½\mathrm{i}\omega}}{\mathrm{i}\omega}$

$$= (2/\omega)\sin(\omega/2) = \frac{\sin(\pi\omega/2\pi)}{\pi\omega/2\pi} = \text{sinc}(\omega/2\pi).$$

b) $\mathcal{F}\{f(t/\alpha)\} = \int_{-\infty}^{\infty} f(t/\alpha)e^{\mathrm{i}\omega t}\mathrm{d}t = \alpha\int_{\mp\infty}^{\pm\infty} f(t')e^{\mathrm{i}\alpha\omega t'}\mathrm{d}t'$ ← change of variable: $t' = t/\alpha$

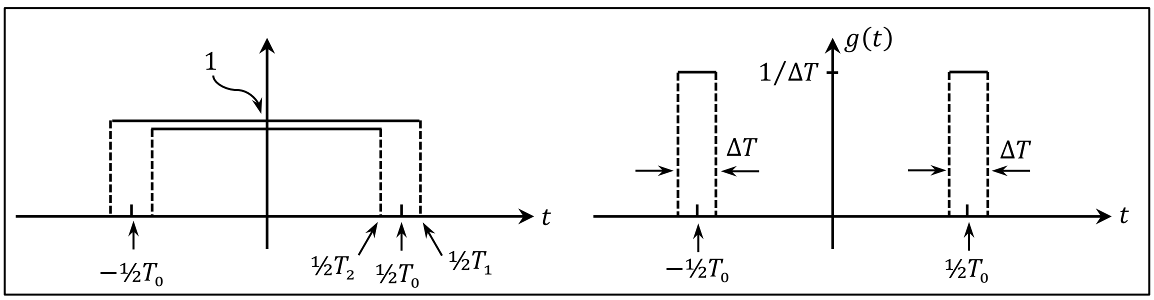
$= |\alpha|\int_{-\infty}^{\infty} f(t')e^{\mathrm{i}\alpha\omega t'}\mathrm{d}t' = |\alpha|F(\alpha\omega).$ ← sign of α is used to change $\int_{+\infty}^{-\infty}\cdots$ to $\int_{-\infty}^{+\infty}\cdots$

c) $\mathcal{F}\{g(t)\} = [T_1\text{sinc}(T_1\omega/2\pi) - T_2\text{sinc}(T_2\omega/2\pi)]/\Delta T$

$= \left[\dfrac{T_1\sin(T_1\omega/2)}{T_1\omega/2} - \dfrac{T_2\sin(T_2\omega/2)}{T_2\omega/2}\right]/\Delta T = \dfrac{2}{\omega\Delta T}[\sin(T_1\omega/2) - \sin(T_2\omega/2)]$

$= \dfrac{4\sin[(T_1-T_2)\omega/4]\cos[(T_1+T_2)\omega/4]}{\omega\Delta T} = \dfrac{4\sin(\omega\Delta T/2)\cos(T_0\omega/2)}{\omega\Delta T}.$

d) The function $g(t)$ is plotted below (on the right-hand side) as the difference between two rectangular pulses having widths T_1 and T_2. In the limit when $\Delta T \to 0$, the function approaches a pair of δ-functions; that is, $g_0(t) = \delta(t - ½T_0) + \delta(t + ½T_0)$.



e) In the limit when $\Delta T \to 0$, the Fourier transform of $g(t)$ computed in part (c) approaches $2\cos(½T_0\omega)$, because $\sin(\omega\Delta T/2) \to \omega\Delta T/2$. Given that $\mathcal{F}\{\delta(t \pm ½T_0)\} = \exp(\mp½\mathrm{i}T_0\omega)$, it is clear that $\mathcal{F}\{g_0(t)\}$ should be the sum of these two exponentials, namely, $2\cos(½T_0\omega)$.

Problem 2) a) $\rho(\boldsymbol{r},t) = \sigma_0\delta(r - R)$. Note that the units of σ_0 are [coulomb/m^2], whereas those of $\rho(\boldsymbol{r},t)$ are [coulomb/m^3]. This is due to the fact that $\delta(r - R)$ has the units of $[1/m]$.

b) $\rho(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\sigma_0\delta(r - R)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t$ (volume of the ring around the k-vector: $2\pi r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r$)

$= \sigma_0\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\mathrm{d}t\int_{r=0}^{\infty}\delta(r - R)\int_{\theta=0}^{\pi}\exp(-\mathrm{i}kr\cos\theta)\,2\pi r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r$

$= (2\pi)^2\sigma_0\delta(\omega)\int_{r=0}^{\infty}\delta(r - R)(r/\mathrm{i}k)\exp(-\mathrm{i}kr\cos\theta)|_{\theta=0}^{\pi}\mathrm{d}r$

$= 4\pi^2\sigma_0\delta(\omega)\int_{r=0}^{\infty}\delta(r - R)(r/\mathrm{i}k)(e^{\mathrm{i}kr} - e^{-\mathrm{i}kr})\mathrm{d}r$

$= 8\pi^2\sigma_0\delta(\omega)\int_{r=0}^{\infty}\delta(r - R)(r/k)\sin(kr)\,\mathrm{d}r$ ← use sifting property of $\delta(r - R)$

$= 8\pi^2\sigma_0 R^2\delta(\omega)\sin(kR)/(kR).$

c) $\psi(\boldsymbol{k},\omega) = \frac{\rho(\boldsymbol{k},\omega)}{\varepsilon_0[k^2-(\omega/c)^2]}$.

d) $\psi(\boldsymbol{r},t) = (2\pi)^{-4}\int_{-\infty}^{\infty}\psi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$

use sifting property of $\delta(\omega)$

$$= (2\pi)^{-4}\int_{-\infty}^{\infty}\frac{8\pi^2\sigma_0 R^2\delta(\omega)\sin(kR)/(kR)}{\varepsilon_0[k^2-(\omega/c)^2]}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$

$$= \frac{\sigma_0 R}{2\pi^2\varepsilon_0}\int_{-\infty}^{\infty}k^{-3}\sin(kR)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{k}$$

volume of the ring around the r-vector

$$= \frac{\sigma_0 R}{2\pi^2\varepsilon_0}\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}k^{-3}\sin(kR)\exp(\mathrm{i}kr\cos\theta)\,\underbrace{2\pi k^2\sin\theta\,\mathrm{d}\theta\mathrm{d}k}$$

$$= \frac{\sigma_0 R}{\pi\varepsilon_0}\int_{k=0}^{\infty}k^{-2}\sin(kR)\int_{\theta=0}^{\pi}k\sin\theta\exp(\mathrm{i}kr\cos\theta)\,\mathrm{d}\theta\mathrm{d}k$$

$$= \frac{\mathrm{i}\sigma_0 R}{\pi\varepsilon_0 r}\int_{k=0}^{\infty}k^{-2}\sin(kR)\exp(\mathrm{i}kr\cos\theta)|_{\theta=0}^{\pi}\mathrm{d}k$$

$$= \frac{2\sigma_0 R}{\pi\varepsilon_0 r}\int_{k=0}^{\infty}k^{-2}\sin(kR)\sin(kr)\,\mathrm{d}k \leftarrow \text{G\&R } \mathbf{3.741}\text{-3}$$

$$= \frac{2\sigma_0 R}{\pi\varepsilon_0 r}\begin{cases}\pi r/2; & r\le R\\ \pi R/2; & r\ge R\end{cases} = \begin{cases}\sigma_0 R/\varepsilon_0; & r\le R,\\ \sigma_0 R^2/(\varepsilon_0 r); & r\ge R.\end{cases}$$

e) $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t) - \partial\boldsymbol{A}(\boldsymbol{r},t)/\partial t = \begin{cases}0; & r<R,\\ \sigma_0 R^2\hat{\boldsymbol{r}}/(\varepsilon_0 r^2); & r>R.\end{cases}$

f) The E-field inside the charged spherical shell is seen to be zero, whereas that outside the shell is $4\pi R^2\sigma_0\hat{\boldsymbol{r}}/(4\pi\varepsilon_0 r^2) = Q\hat{\boldsymbol{r}}/(4\pi\varepsilon_0 r^2)$, where Q is the total charge content of the sphere. The discontinuity of the perpendicular E-field at the sphere's surface, where $r=R$, is σ_0/ε_0, in agreement with Maxwell's boundary condition.

Problem 3) a) In the absence of all four sources, Maxwell's equations for the EM fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ become

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}} \quad\rightarrow\quad \boldsymbol{\nabla}\cdot\boldsymbol{E} = 0, \tag{1}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial\boldsymbol{D}/\partial t \quad\rightarrow\quad \boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\varepsilon_0\,\partial\boldsymbol{E}/\partial t, \tag{2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{B}/\partial t, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0. \tag{4}$$

b) The vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ is defined as a vector field whose curl equals the B-field; that is, $\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{B}(\boldsymbol{r},t)$. Given that the divergence of the curl of any vector field always equals zero, the preceding definition yields: $\boldsymbol{\nabla}\cdot\boldsymbol{B} = \boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t)] = 0$. Consequently, this choice of the vector potential automatically satisfies Maxwell's 4th equation.

The 3rd of Maxwell's equations now becomes $\boldsymbol{\nabla}\times\boldsymbol{E} + \partial\boldsymbol{B}/\partial t = \boldsymbol{\nabla}\times(\boldsymbol{E}+\partial\boldsymbol{A}/\partial t) = 0$. It is seen that $\boldsymbol{E}+\partial\boldsymbol{A}/\partial t$ is a curl-free field. Since the curl of the gradient of any scalar field always vanishes, it must be clear that $\boldsymbol{E}+\partial\boldsymbol{A}/\partial t$ can be equated with the gradient of some (heretofore unknown) scalar field $\psi(\boldsymbol{r},t)$.

Traditionally, $\boldsymbol{E}+\partial\boldsymbol{A}/\partial t$ has been equated with $-\boldsymbol{\nabla}\psi(\boldsymbol{r},t)$, which is, of course, acceptable, considering that the minus sign thus introduced does *not* alter the required vanishing of the curl of $\boldsymbol{E}+\partial\boldsymbol{A}/\partial t$. One thus writes $\boldsymbol{E}+\partial\boldsymbol{A}/\partial t=-\boldsymbol{\nabla}\psi$, and proceeds to express the E-field in terms of the scalar potential ψ and the vector potential $\boldsymbol{A}$ as $\boldsymbol{E}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\psi-\partial\boldsymbol{A}/\partial t$. By construction, this equation, in conjunction with the identity $\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}$, satisfies Maxwell's 3rd equation.

c) Substituting the above $\boldsymbol{E}(\boldsymbol{r},t)$ in Maxwell's (source-free) 1st equation, $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$, we find

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi)+\partial(\boldsymbol{\nabla}\cdot\boldsymbol{A})/\partial t=0. \tag{5}$$

Similarly, substituting in Maxwell's 2nd source-free equation for $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ in terms of the potentials, and also recalling that $\mu_0\varepsilon_0=1/c^2$, one arrives at

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\mu_0\varepsilon_0\frac{\partial(-\boldsymbol{\nabla}\psi-\partial\boldsymbol{A}/\partial t)}{\partial t} \quad\rightarrow\quad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})+\frac{1}{c^2}\left[\frac{\partial^2\boldsymbol{A}}{\partial t^2}+\boldsymbol{\nabla}\left(\frac{\partial\psi}{\partial t}\right)\right]=0. \tag{6}$$

Equations (5) and (6) are the coupled pair of partial differential equations that relate the (source-free) scalar and vector potentials to each other.

d) The Lorenz gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A}+c^{-2}(\partial\psi/\partial t)=0$ may now be used to decouple Eqs.(5) and (6). In the case of Eq.(5), we replace $\boldsymbol{\nabla}\cdot\boldsymbol{A}$ with $-c^{-2}(\partial\psi/\partial t)$, and in the case of Eq.(6), we substitute $-c^2\boldsymbol{\nabla}\cdot\boldsymbol{A}$ for $\partial\psi/\partial t$. We thus find

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi)=\frac{\partial^2\psi(\boldsymbol{r},t)}{c^2\partial t^2}. \tag{7}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\frac{\partial^2\boldsymbol{A}(\boldsymbol{r},t)}{c^2\partial t^2}. \tag{8}$$

These are the decoupled partial differential equations for $\psi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$, respectively.

e) In the Fourier domain, the $\boldsymbol{\nabla}$ operator becomes $\mathrm{i}\boldsymbol{k}$, while the $\partial/\partial t$ operator changes to $-\mathrm{i}\omega$. Thus, Eq.(7), Eq.(8), and the aforementioned Lorenz gauge equation become

$$\mathrm{i}\boldsymbol{k}\cdot\mathrm{i}\boldsymbol{k}\psi(\boldsymbol{k},\omega)=c^{-2}(-\mathrm{i}\omega)^2\psi(\boldsymbol{k},\omega) \quad\rightarrow\quad [k^2-(\omega/c)^2]\psi(\boldsymbol{k},\omega)=0. \tag{9}$$

$$\mathrm{i}\boldsymbol{k}[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k},\omega)]-\mathrm{i}\boldsymbol{k}\times[\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}(\boldsymbol{k},\omega)]=c^{-2}(-\mathrm{i}\omega)^2\boldsymbol{A}(\boldsymbol{k},\omega)$$

$$\boxed{\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}}$$

$$\rightarrow\quad -\cancel{(\boldsymbol{k}\cdot\boldsymbol{A})\boldsymbol{k}}+\cancel{(\boldsymbol{k}\cdot\boldsymbol{A})\boldsymbol{k}}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{A}=-(\omega/c)^2\boldsymbol{A} \quad\rightarrow\quad [k^2-(\omega/c)^2]\boldsymbol{A}(\boldsymbol{k},\omega)=0. \tag{10}$$

$$\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k},\omega)+c^{-2}(-\mathrm{i}\omega)\psi(\boldsymbol{k},\omega)=0 \quad\rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k},\omega)-(\omega/c^2)\psi(\boldsymbol{k},\omega)=0. \tag{11}$$

f) According to Eq.(11), the Lorenz gauge requires the projection onto $\boldsymbol{k}$ of $\boldsymbol{A}(\boldsymbol{k},\omega)$, commonly written as $\boldsymbol{A}_\parallel(\boldsymbol{k},\omega)$, to satisfy the following relation with $\psi(\boldsymbol{k},\omega)$: $\boldsymbol{A}_\parallel=\omega\psi\hat{\boldsymbol{k}}/(c^2k)$. However, such a constraint on $\boldsymbol{A}_\parallel$ in no way modifies, or otherwise affects, the original definition of $\boldsymbol{A}(\boldsymbol{r},t)$ as a vector field whose curl must equal $\boldsymbol{B}(\boldsymbol{r},t)$. The reason is that, in the Fourier domain, the vector potential must relate to the B-field in the following way:

$$\boldsymbol{B}(\boldsymbol{k},\omega)=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}(\boldsymbol{k},\omega)=\mathrm{i}\boldsymbol{k}\times(\boldsymbol{A}_\parallel+\boldsymbol{A}_\perp)=\underbrace{\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\parallel}_{0}+\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\perp=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_\perp. \tag{12}$$

Clearly, it is only the projection of $\boldsymbol{A}(\boldsymbol{k},\omega)$ in a plane perpendicular to $\boldsymbol{k}$ that is needed to specify the B-field. As such, the projection of $\boldsymbol{A}(\boldsymbol{k},\omega)$ onto $\boldsymbol{k}$, which is constrained by the Lorenz gauge, plays no role in the original definition of $\boldsymbol{A}(\boldsymbol{r},t)$ in connection with the B-field.

g) According to Eqs.(9) and (10), both $\psi(\boldsymbol{k},\omega)$ and $\boldsymbol{A}(\boldsymbol{k},\omega)$ will be zero unless $k^2 = (\omega/c)^2$. Since the squared length of the vector $\boldsymbol{k}$ is given by $\boldsymbol{k}\cdot\boldsymbol{k} = k^2$, one concludes that the only way to have nonzero solutions for $\psi(\boldsymbol{k},\omega)$ and $\boldsymbol{A}(\boldsymbol{k},\omega)$ is to demand that EM plane-waves in free space satisfy the condition $|\boldsymbol{k}| = \omega/c$.

h) Transforming $\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t)$ to the Fourier domain yields $\boldsymbol{B}(\boldsymbol{k},\omega) = \mathrm{i}\boldsymbol{k}\times\boldsymbol{A}(\boldsymbol{k},\omega)$, which shows that $\boldsymbol{B}(\boldsymbol{k},\omega)$ must be located in a plane perpendicular to the k-vector.

i) Transforming $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t) - \partial\boldsymbol{A}(\boldsymbol{r},t)/\partial t$ to the Fourier domain yields $\boldsymbol{E}(\boldsymbol{k},\omega) = -\mathrm{i}\boldsymbol{k}\psi(\boldsymbol{k},\omega) + \mathrm{i}\omega\boldsymbol{A}(\boldsymbol{k},\omega)$. Upon substitution for $\psi(\boldsymbol{k},\omega)$ from the Lorenz gauge equation, Eq.(11), we find

$$\boldsymbol{E}(\boldsymbol{k},\omega) = \mathrm{i}\omega\{\boldsymbol{A}(\boldsymbol{k},\omega) - [(c\boldsymbol{k}/\omega)\cdot\boldsymbol{A}(\boldsymbol{k},\omega)](c\boldsymbol{k}/\omega)\}. \tag{13}$$

It was shown in part (g) that, for plane-waves in free space, $|\boldsymbol{k}| = \omega/c$. Consequently, $c\boldsymbol{k}/\omega = \hat{\boldsymbol{k}}$ (i.e., the unit-vector along $\boldsymbol{k}$), and $[\hat{\boldsymbol{k}}\cdot\boldsymbol{A}(\boldsymbol{k},\omega)]\hat{\boldsymbol{k}} = \boldsymbol{A}_{\parallel}(\boldsymbol{k},\omega)$, which is the component of $\boldsymbol{A}(\boldsymbol{k},\omega)$ that is parallel to $\boldsymbol{k}$. We may now write Eq.(13) in simplified form, as follows:

$$\boldsymbol{E}(\boldsymbol{k},\omega) = \mathrm{i}\omega[\boldsymbol{A}(\boldsymbol{k},\omega) - \boldsymbol{A}_{\parallel}(\boldsymbol{k},\omega)] = \mathrm{i}\omega\boldsymbol{A}_{\perp}(\boldsymbol{k},\omega). \tag{14}$$

Here, $\boldsymbol{A}_{\perp}(\boldsymbol{k},\omega)$ is the projection of $\boldsymbol{A}(\boldsymbol{k},\omega)$ in a plane perpendicular to $\boldsymbol{k}$. It is thus seen that $\boldsymbol{E}(\boldsymbol{k},\omega)$ is orthogonal to $\boldsymbol{k}$. Moreover, since $\boldsymbol{B}(\boldsymbol{k},\omega) = \mathrm{i}\boldsymbol{k}\times[\boldsymbol{A}_{\parallel}(\boldsymbol{k},\omega) + \boldsymbol{A}_{\perp}(\boldsymbol{k},\omega)] = \mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_{\perp}(\boldsymbol{k},\omega)$, we conclude that $\boldsymbol{E}(\boldsymbol{k},\omega)$ and $\boldsymbol{B}(\boldsymbol{k},\omega)$ are also orthogonal to each other.

j) Substitution for $\boldsymbol{A}_{\perp}(\boldsymbol{k},\omega)$ from Eq.(14) into the preceding equation for $\boldsymbol{B}(\boldsymbol{k},\omega)$ now yields

$$\boldsymbol{B}(\boldsymbol{k},\omega) = \mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_{\perp}(\boldsymbol{k},\omega) = \mathrm{i}(\omega/c)\hat{\boldsymbol{k}}\times[\boldsymbol{E}(\boldsymbol{k},\omega)/\mathrm{i}\omega] = \hat{\boldsymbol{k}}\times\boldsymbol{E}(\boldsymbol{k},\omega)/c. \tag{15}$$

It is seen that $\boldsymbol{B}(\boldsymbol{k},\omega)$, which is orthogonal to both $\hat{\boldsymbol{k}}$ and $\boldsymbol{E}(\boldsymbol{k},\omega)$, has the same magnitude as $\boldsymbol{E}(\boldsymbol{k},\omega)/c$. Needless to say, cross-multiplication of $\boldsymbol{E}(\boldsymbol{k},\omega)$ into $\hat{\boldsymbol{k}}$ is tantamount to a 90° rotation of $\boldsymbol{E}(\boldsymbol{k},\omega)$ around $\hat{\boldsymbol{k}}$.

k) In general, $\boldsymbol{B}(\boldsymbol{k},\omega) = \mu_0\boldsymbol{H}(\boldsymbol{k},\omega) + \boldsymbol{M}(\boldsymbol{k},\omega)$. In the absence of the magnetization $\boldsymbol{M}$, the H-field should be equal to the B-field divided by μ_0. Therefore, $\boldsymbol{H}(\boldsymbol{k},\omega) = \hat{\boldsymbol{k}}\times\boldsymbol{E}(\boldsymbol{k},\omega)/(\mu_0 c) = \hat{\boldsymbol{k}}\times\boldsymbol{E}(\boldsymbol{k},\omega)/Z_0$. The plane-wave's H-field amplitude is thus obtained by a 90° rotation of $\boldsymbol{E}(\boldsymbol{k},\omega)$ around $\hat{\boldsymbol{k}}$, followed by division by Z_0.
