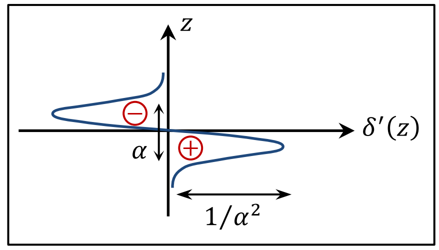
Opti 501 2nd **Midterm Solutions** (11/5/2020) **Time: 75 minutes**

Problem 1) a) $\boldsymbol{P}(\boldsymbol{r},t)=p_0\delta(\boldsymbol{r})\hat{\boldsymbol{z}} \quad \rightarrow \quad \rho_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)=-p_0\delta(x)\delta(y)\delta'(z)$.

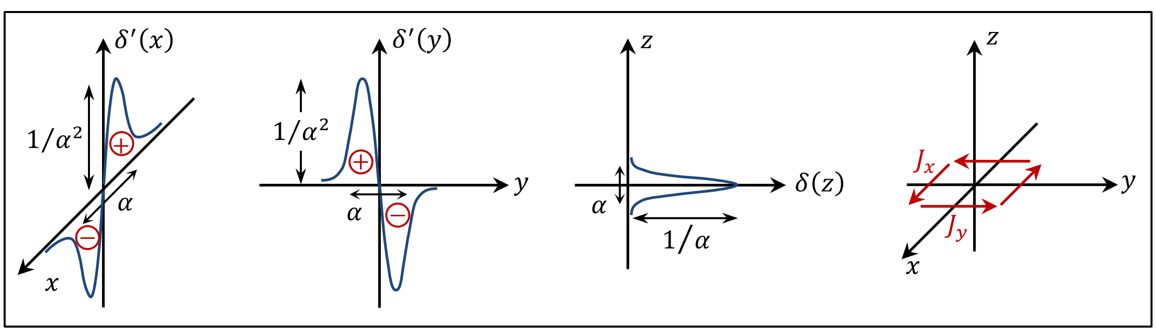


The figure shows a typical plot of $\delta'(z)$ for a sufficiently small value of the parameter α. The function is positive below, and negative above, the origin. The minus sign in the above expression of $\rho_{\text{bound}}^{(\text{e})}$ reverses the sign of the function, so that the (bound) charge-density below the origin is negative, while that above the origin is positive. This is consistent with our understanding that the dipole moment $p_0\hat{\boldsymbol{z}}$ represents a pair of positive and negative charges, with the negative charge slightly below, and the positive charge slightly above, the origin. The area under each lobe of $\delta'(z)$ is $1/\alpha$, making the magnitude of the pair of charges equal to $\pm p_0/\alpha$. Given that the charges are separated by a distance α along the z-axis, as shown in the figure, their dipole moment is $(p_0/\alpha)\alpha\hat{\boldsymbol{z}}=p_0\hat{\boldsymbol{z}}$.

b) $\boldsymbol{M}(\boldsymbol{r},t)=m_0\delta(\boldsymbol{r})\hat{\boldsymbol{z}} \quad \rightarrow \quad \boldsymbol{J}_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t)=\mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)=\mu_0^{-1}\left(\frac{\partial M_z}{\partial y}\hat{\boldsymbol{x}}-\frac{\partial M_z}{\partial x}\hat{\boldsymbol{y}}\right)$

$$=(m_0/\mu_0)[\delta(x)\delta'(y)\delta(z)\hat{\boldsymbol{x}}-\delta'(x)\delta(y)\delta(z)\hat{\boldsymbol{y}}].$$

The figure below shows typical plots of $\delta'(x)$, $\delta'(y)$, and $\delta(z)$ for sufficiently small values of the α parameter. The current-density $\boldsymbol{J}_{\text{bound}}^{(\text{e})}$ has components along the x and y axes, as shown in the rightmost panel of the figure. The direction of the current along each of the four legs of the loop is determined by whether the corresponding lobe of $\delta'(x)$ or $\delta'(y)$ is positive or negative. Each triple-delta-function-product in the above expression of $\boldsymbol{J}_{\text{bound}}^{(\text{e})}$ has a magnitude of $1/\alpha^4$. Integration over each leg's cross-sectional area (α^2) yields the loop current as $I_0=m_0/(\mu_0\alpha^2)$, and multiplication by the loop area $A=\alpha^2$ yields $I_0A=m_0/\mu_0$. The magnetic dipole moment m_0 along the z-axis is thus seen to be the product of μ_0, the loop current I_0, and the loop area A.



c) Invoking the sifting properties of $\delta(\cdot)$ and $\delta'(\cdot)$, direct Fourier transformation of $\rho_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t)$ and $\boldsymbol{J}_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t)$ yields

Sifting property: $\int_{-\infty}^{\infty}f(\zeta)\delta(\zeta)\mathrm{d}\zeta=f(0)$ and $\int_{-\infty}^{\infty}f(\zeta)\delta'(\zeta)\mathrm{d}\zeta=-f'(0)$.

$$\rho_{\text{bound}}^{(\text{e})}(\boldsymbol{k},\omega)=-p_0\int_{-\infty}^{\infty}\delta(x)\delta(y)\delta'(z)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t=-\mathrm{i}2\pi p_0k_z\delta(\omega).$$

$$\boldsymbol{J}_{\text{bound}}^{(\text{e})}(\boldsymbol{k},\omega)=(m_0/\mu_0)\int_{-\infty}^{\infty}[\delta(x)\delta'(y)\delta(z)\hat{\boldsymbol{x}}-\delta'(x)\delta(y)\delta(z)\hat{\boldsymbol{y}}]\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t$$

$$=\mathrm{i}2\pi(m_0/\mu_0)(k_y\hat{\boldsymbol{x}}-k_x\hat{\boldsymbol{y}})\delta(\omega).$$

Alternatively, one could first evaluate the Fourier transforms of $\boldsymbol{P}(\boldsymbol{r},t)$ and $\boldsymbol{M}(\boldsymbol{r},t)$, then calculate the Fourier transforms of the bound charge and current densities, as follows:

$$\boldsymbol{P}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty} p_0\delta(\boldsymbol{r})\hat{\boldsymbol{z}}\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t = 2\pi p_0\delta(\omega)\hat{\boldsymbol{z}}.$$

$$\boldsymbol{M}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty} m_0\delta(\boldsymbol{r})\hat{\boldsymbol{z}}\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t = 2\pi m_0\delta(\omega)\hat{\boldsymbol{z}}.$$

$$\rho_{\text{bound}}^{(\text{e})}(\boldsymbol{k},\omega) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega) = -\mathrm{i}\boldsymbol{k}\cdot 2\pi p_0\delta(\omega)\hat{\boldsymbol{z}} = -\mathrm{i}2\pi p_0 k_z\delta(\omega).$$

$$\boldsymbol{J}_{\text{bound}}^{(\text{e})}(\boldsymbol{k},\omega) = \mathrm{i}\boldsymbol{k}\times\mu_0^{-1}\boldsymbol{M}(\boldsymbol{k},\omega) = \mathrm{i}\boldsymbol{k}\times 2\pi(m_0/\mu_0)\delta(\omega)\hat{\boldsymbol{z}} = \mathrm{i}2\pi(m_0/\mu_0)(k_y\hat{\boldsymbol{x}}-k_x\hat{\boldsymbol{y}})\delta(\omega).$$

Problem 2) a) In the limit when $\omega_0 \to 0$, the oscillating current $I_0\cos(\omega_0 t)$ approaches the constant current I_0. In this limit,

$$\sin(\omega_0 t) \to \omega_0 t \to 0; \qquad \cos(\omega_0 t) \to 1 - ½(\omega_0 t)^2 \to 1;$$

$$J_0(\rho\omega_0/c) \to 1; \qquad J_1(\rho\omega_0/c) \to \rho\omega_0/2c;$$

$$Y_0(\rho\omega_0/c) \to (2/\pi)[\mathcal{C} + \ln(\rho\omega_0/2c)]; \qquad Y_1(\rho\omega_0/c) \to -2c/(\pi\rho\omega_0).$$

Keeping in mind that $lim_{x\to 0}(x\ln x) = 0$, substitution into the expressions of the radiated $\boldsymbol{E}$ and $\boldsymbol{H}$ fields now yields

$$\boldsymbol{E}(\boldsymbol{r},t) \to 0; \qquad \boldsymbol{H}(\boldsymbol{r},t) \to (I_0/2\pi\rho)\hat{\boldsymbol{\varphi}}.$$

b) In the far field, where $\rho\omega_0/c = 2\pi\rho/\lambda_0 \gg 1$ (λ_0 being the vacuum wavelength), one may write

$$J_0(\rho\omega_0/c) \cong \sqrt{2c/(\pi\rho\omega_0)}\cos[\rho\omega_0/c-(\pi/4)]; \quad J_1(\rho\omega_0/c) \cong \sqrt{2c/(\pi\rho\omega_0)}\cos[\rho\omega_0/c-(3\pi/4)];$$

$$Y_0(\rho\omega_0/c) \cong \sqrt{2c/(\pi\rho\omega_0)}\sin[\rho\omega_0/c-(\pi/4)]; \quad Y_1(\rho\omega_0/c) \cong \sqrt{2c/(\pi\rho\omega_0)}\sin[\rho\omega_0/c-(3\pi/4)].$$

Substitution into the expressions of the radiated $\boldsymbol{E}$ and $\boldsymbol{H}$ fields yields

$Z_0 = \sqrt{\mu_0/\varepsilon_0}$ →

$$\boldsymbol{E}(\boldsymbol{r},t) \cong -½\left(Z_0 I_0/\sqrt{\lambda_0\rho}\right)\cos[\omega_0(t-\rho/c)+(\pi/4)]\,\hat{\boldsymbol{z}},$$

← $\lambda_0 = 2\pi c/\omega_0$

$$\boldsymbol{H}(\boldsymbol{r},t) \cong ½\left(I_0/\sqrt{\lambda_0\rho}\right)\cos[\omega_0(t-\rho/c)+(\pi/4)]\,\hat{\boldsymbol{\varphi}}.$$

These functions have the correct retarded form, with $\varphi_0 = \pi/4$. The Poynting vector in the far field is readily seen to be aligned with $\hat{\boldsymbol{\rho}}$ and inversely proportional to ρ, as follows:

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t) \cong ¼(Z_0 I_0^2/\lambda_0\rho)\cos^2[\omega_0(t-\rho/c)+(\pi/4)]\,\hat{\boldsymbol{\rho}}.$$

Problem 3) In the Fourier domain, the Lorenz gauge equation is $\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k},\omega)-(\omega/c^2)\psi(\boldsymbol{k},\omega)=0$. Considering that $\boldsymbol{A}(\boldsymbol{k},\omega)=\mu_0\boldsymbol{J}(\boldsymbol{k},\omega)/[k^2-(\omega/c)^2]$, $\psi(\boldsymbol{k},\omega)=\rho(\boldsymbol{k},\omega)/\varepsilon_0[k^2-(\omega/c)^2]$, and $c^2=(\mu_0\varepsilon_0)^{-1}$, substitution into the Lorenz gauge equation yields $\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{k},\omega)-\omega\rho(\boldsymbol{k},\omega)=0$, which is the expression of charge-current continuity in the Fourier domain.

Problem 4) a) Given the charge-density $\rho(\boldsymbol{r},t)=\rho_0[\text{sphere}(r/R_2)-\text{sphere}(r/R_1)]$, we have

$$\rho(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\rho(\boldsymbol{r},t)e^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\mathrm{d}\boldsymbol{r}\mathrm{d}t = 2\pi\delta(\omega)\rho_0\int_{r=R_1}^{R_2}\int_{\varphi=0}^{\pi}e^{-\mathrm{i}kr\cos\varphi}2\pi r^2\sin\varphi\,\mathrm{d}r\mathrm{d}\varphi$$

integration by parts ↓

$$= \frac{8\pi^2\delta(\omega)\rho_0}{k}\int_{R_1}^{R_2}r\sin(kr)\,\mathrm{d}r = \frac{8\pi^2\delta(\omega)\rho_0}{k}\left[-(r/k)\cos(kr)\big|_{r=R_1}^{R_2}+k^{-1}\int_{R_1}^{R_2}\cos(kr)\,\mathrm{d}r\right]$$

$$= 8\pi^2\delta(\omega)\rho_0\left[\frac{\sin(R_2k) - R_2k\cos(R_2k)}{k^3} - \frac{\sin(R_1k) - R_1k\cos(R_1k)}{k^3}\right]. \quad (1)$$

b) $$\psi(\boldsymbol{k},\omega) = \frac{\rho(\boldsymbol{k},\omega)}{\varepsilon_0[k^2-(\omega/c)^2]} = \frac{8\pi^2\delta(\omega)\rho_0}{\varepsilon_0[k^2-(\omega/c)^2]}\left[\frac{\sin(R_2k) - R_2k\cos(R_2k)}{k^3} - \frac{\sin(R_1k) - R_1k\cos(R_1k)}{k^3}\right]. \quad (2)$$

c) $$\psi(\boldsymbol{r},t) = (2\pi)^{-4}\int_{-\infty}^{\infty}\psi(\boldsymbol{k},\omega)e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\mathrm{d}\boldsymbol{k}\mathrm{d}\omega = (2\pi)^{-4}\int_{-\infty}^{\infty}\frac{\rho(\boldsymbol{k},\omega)}{\varepsilon_0[k^2-(\omega/c)^2]}e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$

$$= \frac{\rho_0}{2\pi^2\varepsilon_0}\int_{k=0}^{\infty}\int_{\varphi=0}^{\pi}\left[\frac{\sin(R_2k)-R_2k\cos(R_2k)}{k^5} - \frac{\sin(R_1k)-R_1k\cos(R_1k)}{k^5}\right]e^{\mathrm{i}kr\cos\varphi}2\pi k^2\sin\varphi\,\mathrm{d}k\mathrm{d}\varphi$$

$$= \frac{2\rho_0}{\pi\varepsilon_0 r}\int_{k=0}^{\infty}\left[\frac{\sin(R_2k)-R_2k\cos(R_2k)}{k^4} - \frac{\sin(R_1k)-R_1k\cos(R_1k)}{k^4}\right]\sin(kr)\,\mathrm{d}k$$

$$= \frac{2\rho_0}{\pi\varepsilon_0 r}\begin{cases}\pi r(3R_2^2 - r^2)/12; & (r \le R_2)\\ \pi R_2^3/6; & (r \ge R_2)\end{cases} - \frac{2\rho_0}{\pi\varepsilon_0 r}\begin{cases}\pi r(3R_1^2 - r^2)/12; & (r \le R_1)\\ \pi R_1^3/6; & (r \ge R_1).\end{cases}$$

Consequently,

$$\psi(\boldsymbol{r},t) = \frac{\rho_0}{2\varepsilon_0}\begin{cases}R_2^2 - R_1^2; & r \le R_1,\\ R_2^2 - ⅓r^2 - ⅔R_1^3/r; & R_1 \le r \le R_2,\\ ⅔(R_2^3 - R_1^3)/r; & r \ge R_2.\end{cases} \quad (3)$$

d) The E-field inside and outside the shell is obtained from the above scalar potential, as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t) = -(\partial\psi/\partial r)\hat{\boldsymbol{r}} = \frac{\rho_0\hat{\boldsymbol{r}}}{3\varepsilon_0 r^2}\begin{cases}0; & r \le R_1,\\ r^3 - R_1^3; & R_1 \le r \le R_2,\\ R_2^3 - R_1^3; & r \ge R_2.\end{cases} \quad (4)$$

e) Average E_r within the shell's wall $= \left(\int_{R_1}^{R_2}E_r\mathrm{d}r\right)/(R_2 - R_1)$

$$= \frac{\rho_0}{3\varepsilon_0(R_2-R_1)}\int_{R_1}^{R_2}\left(r - \frac{R_1^3}{r^2}\right)\mathrm{d}r = \frac{\rho_0}{3\varepsilon_0(R_2-R_1)}\left[½(R_2^2 - R_1^2) + R_1^3\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\right]$$

$$= \frac{\rho_0(R_2^3 - R_2R_1^2 + 2R_1^3 - 2R_2R_1^2)}{6\varepsilon_0(R_2 - R_1)R_2} = \frac{\rho_0[(R_2^2 - R_1^2)R_2 - 2R_1^2(R_2 - R_1)]}{6\varepsilon_0(R_2 - R_1)R_2} = \frac{\rho_0(R_2 - R_1)^2(R_2+2R_1)}{6\varepsilon_0(R_2 - R_1)R_2}$$

$$= \frac{\rho_0(R_2 - R_1)(R_2 + 2R_1)}{6\varepsilon_0 R_2}. \quad (5)$$

f) The ratio of the average E_r within the shell's wall to the E-field at $r = R_2^+$ is given by

$$\frac{\text{average } E_r}{E_r \text{ immediately outside the shell}} = \frac{\rho_0(R_2 - R_1)(R_2 + 2R_1)/6\varepsilon_0 R_2}{\rho_0(R_2^3 - R_1^3)/3\varepsilon_0 R_2^2} = \frac{R_2(R_2 + 2R_1)}{2(R_2^2 + R_2R_1 + R_1^2)}. \quad (6)$$

The above ratio approaches ½ when $R_1 \to R_2$.
