Opti 501 **2nd Midterm Solutions** (11/7/2019) **Time: 75 minutes**

Problem 1) a) $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) = J_{s0}\text{Rect}(x/W)\text{Rect}(y/L)\delta(z)\sin(\omega_0 t)\,\hat{\boldsymbol{y}}$.

b) $\boldsymbol{\nabla}\cdot\boldsymbol{J} + \partial\rho/\partial t = 0$

$\rightarrow\ \partial\rho/\partial t = -\boldsymbol{\nabla}\cdot\boldsymbol{J} = -\partial J_y/\partial y = -J_{s0}\text{Rect}(x/W)[\delta(y+½L) - \delta(y-½L)]\delta(z)\sin(\omega_0 t)$

$\rightarrow\quad \rho_{\text{free}}(\boldsymbol{r},t) = (J_{s0}/\omega_0)\text{Rect}(x/W)[\delta(y+½L) - \delta(y-½L)]\delta(z)\cos(\omega_0 t)$.

c) Charges appear only at the front-edge ($y = L/2$) and rear edge ($y = -L/2$) of the conductor; their linear density is J_{s0}/ω_0 [coulomb/meter], and they oscillate in time as $\cos(\omega_0 t)$. When the charge-density at the front-edge is positive, that at the rear-edge will be negative, and vice-versa. The total charge is, therefore, zero at all times.

Problem 2) a) $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi - \partial\boldsymbol{A}/\partial t = A_0\omega_0\,J_0(\omega_0 r/c)\sin(\omega_0 t)\,\hat{\boldsymbol{z}}$.

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t) = -(\partial A_z/\partial r)\hat{\boldsymbol{\varphi}} = (A_0\omega_0/c)\,J_1(\omega_0 r/c)\cos(\omega_0 t)\,\hat{\boldsymbol{\varphi}}.$$

b) Since the tangential component of the E-field at the inner surface of the hollow cylinder vanishes, the boundary condition associated with $\boldsymbol{E}_\parallel$ is satisfied. The tangential H-field component must be equal in magnitude and perpendicular in direction to the surface current-density at the inner cylindrical surface. Consequently,

$$\boldsymbol{J}_s(t) = -(A_0\omega_0/\mu_0 c)\,J_1(\omega_0 R/c)\cos(\omega_0 t)\,\hat{\boldsymbol{z}}.$$

Note that, since the zeros of $J_0(\cdot)$ do not coincide with those of $J_1(\cdot)$, the H-field at the inner cylindrical surface and, consequently, the surface current $\boldsymbol{J}_s$, do not vanish. Both perpendicular field components $E_r(r=R,\varphi,z,t)$ and $B_r(r=R,\varphi,z,t)$ at the inner surface are zero. The latter confirms that $\boldsymbol{B}_\perp$ satisfies Maxwell's boundary condition at $r = R$, and the former indicates that no electric charges reside on the interior wall of the cylinder. The absence of surface charges is also consistent with the charge-current continuity equation, as $\boldsymbol{\nabla}\cdot\boldsymbol{J}_s(r=R,\varphi,z,t) = 0$.

Problem 3) a) The free current-density is obtained by an inverse Fourier transform, as follows:

$$\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) = (2\pi)^{-4}\int_{-\infty}^{\infty} I_0\delta(k-k_0)[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]\hat{\boldsymbol{k}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\,\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$

$$= (2\pi)^{-4}I_0[\exp(-\mathrm{i}\omega_0 t) - \exp(\mathrm{i}\omega_0 t)]\int_{-\infty}^{\infty}\delta(k-k_0)\hat{\boldsymbol{k}}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{k}$$

$$= -2\mathrm{i}(2\pi)^{-4}I_0\sin(\omega_0 t)\int_{k=0}^{\infty}\int_{\varphi=0}^{\pi}\delta(k-k_0)\cos\varphi\,\hat{\boldsymbol{r}}\exp(\mathrm{i}kr\cos\varphi)\,(2\pi k^2\sin\varphi)\mathrm{d}\varphi\mathrm{d}k$$

$$= -2\mathrm{i}(2\pi)^{-3}I_0\sin(\omega_0 t)\hat{\boldsymbol{r}}\int_{k=0}^{\infty}k^2\delta(k-k_0)\int_{\varphi=0}^{\pi}\sin\varphi\cos\varphi\exp(\mathrm{i}kr\cos\varphi)\mathrm{d}\varphi\,\mathrm{d}k$$

$$= -2\mathrm{i}(2\pi)^{-3}I_0\sin(\omega_0 t)\hat{\boldsymbol{r}}\int_{0}^{\infty}k^2\delta(k-k_0)\frac{2\mathrm{i}\,[\sin(kr) - kr\cos(kr)]}{(kr)^2}\mathrm{d}k$$

$$= \frac{I_0}{2\pi^3 r^2}[\sin(k_0 r) - k_0 r\cos(k_0 r)]\sin(\omega_0 t)\hat{\boldsymbol{r}}. \qquad (1)$$

This spherically symmetric current-density flows in the radial direction $\hat{\boldsymbol{r}}$ and oscillates with frequency ω_0. In the limit when $r \to 0$, we have

$$\sin(k_0 r) - k_0 r\cos(k_0 r) \to \left[k_0 r - \tfrac{1}{3!}(k_0 r)^3 + \cdots\right] - k_0 r\left[1 - \tfrac{1}{2!}(k_0 r)^2 + \cdots\right] = ⅓(k_0 r)^3. \quad (2)$$

Consequently, $\boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) \to 0$ when $r \to 0$.

b) $\omega\rho(\boldsymbol{k}, \omega) = \boldsymbol{k} \cdot \boldsymbol{J}(\boldsymbol{k}, \omega) \;\to\; \rho(\boldsymbol{k}, \omega) = (I_0 k_0/\omega_0)\delta(k - k_0)[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad (3)$

Inverse Fourier transformation now yields

$$\begin{aligned}
\rho_{\text{free}}(\boldsymbol{r}, t) &= \frac{I_0 k_0}{(2\pi)^4\omega_0}\int_{-\infty}^{\infty}\delta(k - k_0)[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\,\mathrm{d}\boldsymbol{k}\mathrm{d}\omega \\
&= \frac{I_0 k_0}{(2\pi)^4\omega_0}[\exp(-\mathrm{i}\omega_0 t) + \exp(\mathrm{i}\omega_0 t)]\int_{-\infty}^{\infty}\delta(k - k_0)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{k} \\
&= \frac{2I_0 k_0\cos(\omega_0 t)}{(2\pi)^4\omega_0}\int_{k=0}^{\infty}\int_{\varphi=0}^{\pi}\delta(k - k_0)\exp(\mathrm{i}kr\cos\varphi)\,(2\pi k^2\sin\varphi)\mathrm{d}\varphi\mathrm{d}k \\
&= \frac{2I_0 k_0\cos(\omega_0 t)}{(2\pi)^3\omega_0}\int_{k=0}^{\infty}k^2\delta(k - k_0)\int_{\varphi=0}^{\pi}\sin\varphi\exp(\mathrm{i}kr\cos\varphi)\,\mathrm{d}\varphi\,\mathrm{d}k \\
&= \frac{2I_0 k_0\cos(\omega_0 t)}{(2\pi)^3\omega_0}\int_{0}^{\infty}k^2\delta(k - k_0)\left.\frac{\exp(\mathrm{i}kr\cos\varphi)}{-\mathrm{i}kr}\right|_{\varphi=0}^{\pi}\mathrm{d}k \\
&= \frac{4I_0 k_0\cos(\omega_0 t)}{(2\pi)^3\omega_0 r}\int_{0}^{\infty}k\sin(kr)\,\delta(k - k_0)\mathrm{d}k = \left(\frac{I_0 k_0^2}{2\pi^3\omega_0 r}\right)\sin(k_0 r)\cos(\omega_0 t). \qquad (4)
\end{aligned}$$

The charge-density is also spherically symmetric and oscillates with frequency ω_0. In the limit when $r \to 0$, we have $\sin(k_0 r)/r \to k_0$. Thus, neither the charge-density nor the current-density have singularities at $r = 0$.

c) To confirm the charge-current continuity equation in the spacetime domain, we derive the free charge-density from the divergence of the current-density, as follows:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) = \frac{\partial(r^2 J_r)}{r^2\partial r} &= \left(\frac{I_0}{2\pi^3 r^2}\right)\frac{\partial}{\partial r}[\sin(k_0 r) - k_0 r\cos(k_0 r)]\sin(\omega_0 t) \\
&= \left(\frac{I_0 k_0^2}{2\pi^3 r}\right)\sin(k_0 r)\sin(\omega_0 t). \qquad (5)
\end{aligned}$$

The continuity equation then yields

$$\begin{aligned}
&\partial\rho_{\text{free}}(\boldsymbol{r}, t)/\partial t = -\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) = -\left(\frac{I_0 k_0^2}{2\pi^3 r}\right)\sin(k_0 r)\sin(\omega_0 t) \\
&\to \quad \rho_{\text{free}}(\boldsymbol{r}, t) = \left(\frac{I_0 k_0^2}{2\pi^3\omega_0 r}\right)\sin(k_0 r)\cos(\omega_0 t). \qquad (6)
\end{aligned}$$

The above equation is seen to be identical to Eq.(4) and, therefore, the charge-current continuity equation is satisfied

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Problem 4) a) Considering that $\hat{\boldsymbol{y}} = \sin\varphi\,\hat{\boldsymbol{r}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}$ in the (r,φ,z) cylindrical coordinate system, we have

$$\boldsymbol{M}(\boldsymbol{r},t) = m_{s0}\mathrm{Circ}(r/R)\delta(z)\cos(\omega_0 t)\,(\sin\varphi\,\hat{\boldsymbol{r}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}).$$

b) $$\rho_{\text{bound}}^{(m)}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r},t) = -\frac{\partial(rM_r)}{r\partial r} - \frac{\partial M_\varphi}{r\partial\varphi}$$

$$= -m_{s0}\frac{\partial[r\,\mathrm{Circ}(r/R)]}{r\partial r}\sin\varphi\,\delta(z)\cos(\omega_0 t) - m_{s0}\mathrm{Circ}(r/R)\frac{\partial\cos\varphi}{r\partial\varphi}\delta(z)\cos(\omega_0 t)$$

$$= -m_{s0}[r^{-1}\mathrm{Circ}(r/R) - \delta(r-R)]\sin\varphi\,\delta(z)\cos(\omega_0 t)$$

$$+m_{s0}\mathrm{Circ}(r/R)r^{-1}\sin\varphi\,\delta(z)\cos(\omega_0 t)$$

$$= m_{s0}\delta(r-R)\sin\varphi\,\delta(z)\cos(\omega_0 t).$$

Bound magnetic charges appear only at the disk's rim (i.e., at $r = R$ and $z = 0$), with the largest magnetic monopole density around $\varphi = \pm 90°$, and no magnetic charges at $\varphi = 0°$ and $180°$. The charges oscillate in time as $\cos(\omega_0 t)$.

$$\boldsymbol{J}_{\text{bound}}^{(m)}(\boldsymbol{r},t) = \partial\boldsymbol{M}(\boldsymbol{r},t)/\partial t = -m_{s0}\omega_0\mathrm{Circ}(r/R)\delta(z)\sin(\omega_0 t)\hat{\boldsymbol{y}}.$$

The bound magnetic current exists everywhere within the disk. The magnetic current-density is aligned with the y-axis and oscillates in time as $\sin(\omega_0 t)$.

c) $$\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t) = 0.$$

$$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t) = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t) = -\frac{\partial M_\varphi}{\mu_0\partial z}\hat{\boldsymbol{r}} + \frac{\partial M_r}{\mu_0\partial z}\hat{\boldsymbol{\varphi}} + \frac{1}{\mu_0 r}\left[\frac{\partial(rM_\varphi)}{\partial r} - \frac{\partial M_r}{\partial\varphi}\right]\hat{\boldsymbol{z}}$$

$$= -\left(\frac{m_{s0}}{\mu_0}\right)\mathrm{Circ}(r/R)\cos\varphi\,\delta'(z)\cos(\omega_0 t)\,\hat{\boldsymbol{r}}$$

$$+\left(\frac{m_{s0}}{\mu_0}\right)\mathrm{Circ}(r/R)\sin\varphi\,\delta'(z)\cos(\omega_0 t)\,\hat{\boldsymbol{\varphi}}$$

$$+\left(\frac{m_{s0}}{\mu_0 r}\right)\left[\frac{\partial[r\mathrm{Circ}(r/R)]}{\partial r}\cos\varphi\,\delta(z)\cos(\omega_0 t) - \mathrm{Circ}(r/R)\cos\varphi\,\delta(z)\cos(\omega_0 t)\right]\hat{\boldsymbol{z}}$$

$$= -\left(\frac{m_{s0}}{\mu_0}\right)[\mathrm{Circ}(r/R)\underbrace{(\cos\varphi\,\hat{\boldsymbol{r}} - \sin\varphi\,\hat{\boldsymbol{\varphi}})}_{\hat{\boldsymbol{x}}}\delta'(z) + \delta(r-R)\cos\varphi\,\delta(z)\hat{\boldsymbol{z}}]\cos(\omega_0 t).$$

d) No electric charges exist in a purely magnetic material. The bound electric current-density resides primarily on the top and bottom facets of the disk, flowing along $\pm x$ directions. These currents, being equal in magnitude and opposite in direction at all times, connect to each other at the disk's rim (i.e., where $r = R$ and $z = 0$). On the rim, the current flows along $\pm z$ directions, nearing its peak value around $\varphi = 0°$ and $180°$; the current drops to zero at $\varphi = \pm 90°$. The bound electric current-density everywhere oscillates in time as $\cos(\omega_0 t)$.
