Opti 501 $\quad$ 2^{nd} Midterm Solutions (10/25/2018) $\quad$ Time: 75 minutes

Problem 1)

a) $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=\rho_{\text{free}}(\boldsymbol{r},t) \quad\rightarrow\quad \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{D}(\boldsymbol{k},\omega)=\rho_{\text{free}}(\boldsymbol{k},\omega).$ (1)

$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\partial_t\boldsymbol{D}(\boldsymbol{r},t) \quad\rightarrow\quad \mathrm{i}\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)=\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\omega\boldsymbol{D}(\boldsymbol{k},\omega).$ (2)

$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial_t\boldsymbol{B}(\boldsymbol{r},t) \quad\rightarrow\quad \mathrm{i}\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega)=\mathrm{i}\omega\boldsymbol{B}(\boldsymbol{k},\omega).$ (3)

$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0 \quad\rightarrow\quad \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{B}(\boldsymbol{k},\omega)=0.$ (4)

b) To eliminate $\boldsymbol{D}$ and $\boldsymbol{B}$ from the above equations, we use the identities $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P}$ and $\boldsymbol{B}=\mu_0\boldsymbol{H}+\boldsymbol{M}$. We will have

i) $\varepsilon_0\boldsymbol{k}\cdot\boldsymbol{E}=-\mathrm{i}\rho_{\text{free}}-\boldsymbol{k}\cdot\boldsymbol{P}.$ (5)

ii) $\boldsymbol{k}\times\boldsymbol{H}=-\mathrm{i}\boldsymbol{J}_{\text{free}}-\omega\varepsilon_0\boldsymbol{E}-\omega\boldsymbol{P}.$ (6)

iii) $\boldsymbol{k}\times\boldsymbol{E}=\omega\mu_0\boldsymbol{H}+\omega\boldsymbol{M}.$ (7)

iv) $\mu_0\boldsymbol{k}\cdot\boldsymbol{H}=-\boldsymbol{k}\cdot\boldsymbol{M}.$ (8)

Next, we cross-multiply equation (ii) into $\boldsymbol{k}$ on the left-hand side, so that, later on, we will be able to substitute for $\boldsymbol{k}\times\boldsymbol{E}$ from equation (iii). We find

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H})=-\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}-\varepsilon_0\omega\boldsymbol{k}\times\boldsymbol{E}-\omega\boldsymbol{k}\times\boldsymbol{P}. \quad (9)$$

The vector identity $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H})=(\boldsymbol{k}\cdot\boldsymbol{H})\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}$ can now be used in conjunction with equations (iii) and (iv) to yield $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H})=-\mu_0^{-1}(\boldsymbol{k}\cdot\boldsymbol{M})\boldsymbol{k}-k^2\boldsymbol{H}$, and, subsequently,

$$-\mu_0^{-1}(\boldsymbol{k}\cdot\boldsymbol{M})\boldsymbol{k}-k^2\boldsymbol{H}=-\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}-\varepsilon_0\omega(\mu_0\omega\boldsymbol{H}+\omega\boldsymbol{M})-\omega\boldsymbol{k}\times\boldsymbol{P}. \quad (10)$$

$$\rightarrow\quad (k^2-\mu_0\varepsilon_0\omega^2)\boldsymbol{H}=\mathrm{i}\boldsymbol{k}\times(\boldsymbol{J}_{\text{free}}-\mathrm{i}\omega\boldsymbol{P})-\mu_0^{-1}(\boldsymbol{k}\cdot\boldsymbol{M})\boldsymbol{k}+\varepsilon_0\omega^2\boldsymbol{M}. \quad (11)$$

$$\rightarrow\quad \boldsymbol{H}(\boldsymbol{k},\omega)=\frac{\mathrm{i}\boldsymbol{k}\times\mu_0[\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\omega\boldsymbol{P}(\boldsymbol{k},\omega)]-[\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)]\boldsymbol{k}+(\omega/c)^2\boldsymbol{M}(\boldsymbol{k},\omega)}{\mu_0[k^2-(\omega/c)^2]}. \quad (12)$$

To find the E-field, we cross-multiply equation (iii) into $\boldsymbol{k}$ on the left-hand side, then substitute for $\boldsymbol{k}\times\boldsymbol{H}$ from equation (ii), to find

$$\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{E}]=\mu_0\omega\boldsymbol{k}\times\boldsymbol{H}+\omega\boldsymbol{k}\times\boldsymbol{M}. \quad (13)$$

The vector identity $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E})=(\boldsymbol{k}\cdot\boldsymbol{E})\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}$ can now be used in conjunction with equations (i) and (ii) to yield $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E})=\varepsilon_0^{-1}(-\mathrm{i}\rho_{\text{free}}-\boldsymbol{k}\cdot\boldsymbol{P})\boldsymbol{k}-k^2\boldsymbol{E}$, and, subsequently,

$$\varepsilon_0^{-1}(-\mathrm{i}\rho_{\text{free}}-\boldsymbol{k}\cdot\boldsymbol{P})\boldsymbol{k}-k^2\boldsymbol{E}=\mu_0\omega(-\mathrm{i}\boldsymbol{J}_{\text{free}}-\varepsilon_0\omega\boldsymbol{E}-\omega\boldsymbol{P})+\omega\boldsymbol{k}\times\boldsymbol{M}. \quad (14)$$

$$\rightarrow\quad (k^2-\mu_0\varepsilon_0\omega^2)\boldsymbol{E}=-\mathrm{i}\varepsilon_0^{-1}(\rho_{\text{free}}-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P})\boldsymbol{k}+\mathrm{i}\mu_0\omega(\boldsymbol{J}_{\text{free}}-\mathrm{i}\omega\boldsymbol{P}+\mathrm{i}\mu_0^{-1}\boldsymbol{k}\times\boldsymbol{M}). \quad (15)$$

$$\rightarrow\quad \boldsymbol{E}(\boldsymbol{k},\omega)=\frac{-\mathrm{i}[\rho_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k}+\mathrm{i}\mu_0\varepsilon_0\omega[\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\omega\boldsymbol{P}(\boldsymbol{k},\omega)+\mathrm{i}\boldsymbol{k}\times\mu_0^{-1}\boldsymbol{M}(\boldsymbol{k},\omega)]}{\varepsilon_0[k^2-(\omega/c)^2]}. \quad (16)$$

c) In the final expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ in Eqs.(12) and (16), various sources appear as follows:

Free electric charge-density: $\rho_{\text{free}}(\boldsymbol{k},\omega)$

Free electric current-density: $\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)$

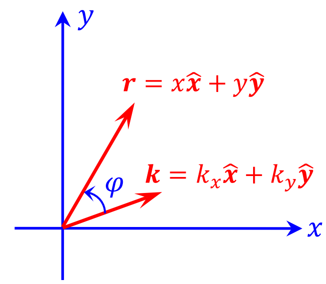
Bound electric charge-density: $-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)$

Bound electric current-density: $-\mathrm{i}\omega\boldsymbol{P}(\boldsymbol{k},\omega)+\mathrm{i}\boldsymbol{k}\times\mu_0^{-1}\boldsymbol{M}(\boldsymbol{k},\omega)$

Bound magnetic charge-density: $-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)$

Bound magnetic current-density: $-\mathrm{i}\omega\boldsymbol{M}(\boldsymbol{k},\omega)$

Problem 2)

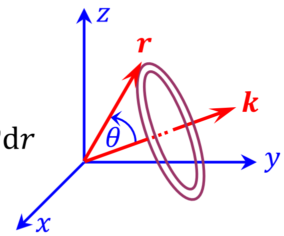


a) $$\mathcal{F}\{\text{circ}(r)\}=\iint_{-\infty}^{\infty}\text{circ}(r)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$$
$$=\int_{r=0}^{1}\int_{\varphi=0}^{2\pi}\exp(-\mathrm{i}kr\cos\varphi)\,r\mathrm{d}r\mathrm{d}\varphi$$
$$=\int_{r=0}^{1}r\left[\int_{\varphi=0}^{2\pi}\exp(-\mathrm{i}kr\cos\varphi)\mathrm{d}\varphi\right]\mathrm{d}r$$
$$=2\pi\int_{r=0}^{1}rJ_0(kr)\mathrm{d}r=(2\pi/k^2)\int_{x=0}^{k}xJ_0(x)\mathrm{d}x$$
$$=(2\pi/k^2)xJ_1(x)|_{x=0}^{k}=2\pi J_1(k)/k. \quad (1)$$

b) $$\mathcal{F}\{\alpha^{-2}\text{circ}(r/\alpha)\}=\alpha^{-2}\iint_{-\infty}^{\infty}\text{circ}(r/\alpha)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$$
$$=\alpha^{-2}\int_{r=0}^{\alpha}\int_{\varphi=0}^{2\pi}\exp(-\mathrm{i}kr\cos\varphi)\,r\mathrm{d}r\mathrm{d}\varphi$$
$$=\alpha^{-2}\int_{r=0}^{\alpha}r\left[\int_{\varphi=0}^{2\pi}\exp(-\mathrm{i}kr\cos\varphi)\mathrm{d}\varphi\right]\mathrm{d}r$$
$$=2\pi\alpha^{-2}\int_{r=0}^{\alpha}rJ_0(kr)\mathrm{d}r=(2\pi/\alpha^2k^2)\int_{x=0}^{\alpha k}xJ_0(x)\mathrm{d}x$$
$$=(2\pi/\alpha^2k^2)xJ_1(x)|_{x=0}^{\alpha k}=2\pi J_1(\alpha k)/(\alpha k). \quad (2)$$

c) In the limit when $\alpha\to0$, both the numerator and denominator of the Fourier transform function appearing on the right-hand side of Eq.(2) approach zero. However, for sufficiently small values of x, we have $J_1(x)\sim x/2$, and, therefore, in the limit when $\alpha\to0$, the Fourier transform of our 2-dimensional δ-function approaches $\pi\alpha k/\alpha k=\pi$, which is the volume under the function $f(x/\alpha,y/\alpha)=\alpha^{-2}\text{circ}(r/\alpha)$ for any and all values of $\alpha>0$.

Problem 3)



a) $$\boldsymbol{M}(\boldsymbol{k},\omega)=\int_{-\infty}^{\infty}\boldsymbol{M}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t$$
$$=2\pi\delta(\omega)\int_{r=0}^{R}\int_{\theta=0}^{\pi}(M_0/R)(r\cos\theta\,\hat{\boldsymbol{k}})\exp(-\mathrm{i}kr\cos\theta)\,2\pi r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r$$
$$=4\pi^2(M_0/R)\delta(\omega)\hat{\boldsymbol{k}}\int_{r=0}^{R}r^3\left[\int_{\theta=0}^{\pi}\sin\theta\cos\theta\exp(-\mathrm{i}kr\cos\theta)\,\mathrm{d}\theta\right]\mathrm{d}r$$

$= 4\pi^2(M_0/R)\delta(\omega)\hat{\boldsymbol{k}}\int_{r=0}^{R} r^3\{-2\mathrm{i}[\sin(kr) - kr\cos(kr)]/(k^2r^2)\}\mathrm{d}r$

$= -\mathrm{i}8\pi^2(M_0/R)\delta(\omega)(\hat{\boldsymbol{k}}/k^3)\int_{r=0}^{R}[kr\sin(kr) - k^2r^2\cos(kr)]\mathrm{d}r$

$= -\mathrm{i}8\pi^2(M_0/R)\delta(\omega)(\hat{\boldsymbol{k}}/k^4)\int_{x=0}^{kR}(x\sin x - x^2\cos x)\mathrm{d}x$

$= \mathrm{i}8\pi^2(M_0/R)[(kR)^2\sin(kR) + 3(kR)\cos(kR) - 3\sin(kR)]\delta(\omega)(\hat{\boldsymbol{k}}/k^4).$

b) The bound electric current-density due to the magnetization is now evaluated as follows:

$$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{k},\omega) = \mathrm{i}\boldsymbol{k}\times\mu_0^{-1}\boldsymbol{M}(\boldsymbol{k},\omega)$$

$$= -\frac{8\pi^2\mu_0^{-1}M_0[(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)]\delta(\omega)}{Rk^4}\boldsymbol{k}\times\hat{\boldsymbol{k}} = 0.$$

c) The magnetic B-field is produced by the total current-density $\boldsymbol{J}_{\text{total}}^{(e)}(\boldsymbol{k},\omega)$, namely,

$$\boldsymbol{B}(\boldsymbol{k},\omega) = \frac{\mathrm{i}\boldsymbol{k}\times\mu_0\boldsymbol{J}_{\text{total}}^{(e)}(\boldsymbol{k},\omega)}{k^2-(\omega/c)^2}.$$

Considering that the current-density in the present problem is zero, the magnetic B-field is zero everywhere (i.e., inside as well as outside our permanently magnetized sphere).

d) In general, $\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)$. Presently, the B-field is zero everywhere. Therefore, the magnetic H-field is zero outside the magnetized sphere, but, inside the sphere, it is given by $\boldsymbol{H}(\boldsymbol{r},t) = -\mu_0^{-1}\boldsymbol{M}(\boldsymbol{r},t) = -M_0\boldsymbol{r}/(\mu_0R)$.

Digression: In part (a), one of the integrals is evaluated by the method of integration-by-parts, as follows:

i) $\int_0^{x_0} x\sin x\,\mathrm{d}x = -x\cos x|_0^{x_0} + \int_0^{x_0}\cos x\,\mathrm{d}x = \sin x_0 - x_0\cos x_0.$

ii) $\int_0^{x_0} x^2\cos x\,\mathrm{d}x = x^2\sin x|_0^{x_0} - 2\int_0^{x_0} x\sin x\,\mathrm{d}x = x_0^2\sin x_0 + 2x_0\cos x_0 - 2\sin x_0.$

Problem 4) Part (a) of this problem is similar to Problem 3(a), with P_0 being substituted for M_0.

a) $\boldsymbol{P}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\boldsymbol{P}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t$

$$= 2\pi\delta(\omega)\int_{r=0}^{R}\int_{\theta=0}^{\pi}(P_0/R)(r\cos\theta\,\hat{\boldsymbol{k}})\exp(-\mathrm{i}kr\cos\theta)\,2\pi r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r$$

$$= 4\pi^2(P_0/R)\delta(\omega)\hat{\boldsymbol{k}}\int_{r=0}^{R} r^3\Big[\int_{\theta=0}^{\pi}\sin\theta\cos\theta\exp(-\mathrm{i}kr\cos\theta)\,\mathrm{d}\theta\Big]\mathrm{d}r$$

$$= 4\pi^2(P_0/R)\delta(\omega)\hat{\boldsymbol{k}}\int_{r=0}^{R} r^3\{-2\mathrm{i}[\sin(kr) - kr\cos(kr)]/(k^2r^2)\}\mathrm{d}r$$

$$= -\mathrm{i}8\pi^2(P_0/R)\delta(\omega)(\hat{\boldsymbol{k}}/k^3)\int_{r=0}^{R}[kr\sin(kr) - k^2r^2\cos(kr)]\mathrm{d}r$$

$$= -\mathrm{i}8\pi^2(P_0/R)\delta(\omega)(\hat{\boldsymbol{k}}/k^4)\int_{x=0}^{kR}(x\sin x - x^2\cos x)\mathrm{d}x$$

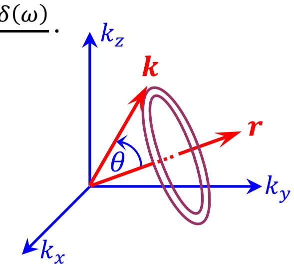
$$= \mathrm{i}8\pi^2(P_0/R)[(kR)^2\sin(kR) + 3(kR)\cos(kR) - 3\sin(kR)]\delta(\omega)(\hat{\boldsymbol{k}}/k^4).$$

b) The bound electric charge-density due to the polarization may now be evaluated, as follows:

$$\rho_{\text{bound}}^{(e)}(\boldsymbol{k},\omega) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega) = \frac{8\pi^2P_0[(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)]\delta(\omega)}{Rk^3}.$$

c) The (static) electric field is given by $\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(\boldsymbol{r})$, which, in the Fourier domain, becomes

$$\boldsymbol{E}(\boldsymbol{k},\omega) = \frac{(-\mathrm{i}\boldsymbol{k})\rho_{\text{bound}}^{(e)}(\boldsymbol{k},\omega)}{\varepsilon_0[k^2-(\omega/c)^2]} = -\frac{\mathrm{i}8\pi^2 P_0[(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)]\boldsymbol{k}\delta(\omega)}{\varepsilon_0 Rk^3[k^2-(\omega/c)^2]}.$$



d) $\boldsymbol{E}(\boldsymbol{r},t) = (2\pi)^{-4}\int_{-\infty}^{\infty}\boldsymbol{E}(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$

$$= -\frac{\mathrm{i}P_0}{2\pi^2\varepsilon_0 R}\iiint_{-\infty}^{\infty}\frac{[(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)]\boldsymbol{k}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}{k^5}\,\mathrm{d}\boldsymbol{k}$$

$$= -\frac{\mathrm{i}P_0}{2\pi^2\varepsilon_0 R}\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\frac{[(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)](k\cos\theta\hat{\boldsymbol{r}})\exp(\mathrm{i}kr\cos\theta)}{k^5}\,2\pi k^2\sin\theta\,\mathrm{d}k\mathrm{d}\theta$$

$$= -\frac{\mathrm{i}P_0\hat{\boldsymbol{r}}}{\pi\varepsilon_0 R}\int_{k=0}^{\infty}\frac{(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)}{k^2}\left[\int_{\theta=0}^{\pi}\sin\theta\cos\theta\exp(\mathrm{i}kr\cos\theta)\,\mathrm{d}\theta\right]\mathrm{d}k$$

$$= -\frac{\mathrm{i}P_0\hat{\boldsymbol{r}}}{\pi\varepsilon_0 R}\int_{0}^{\infty}\frac{(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)}{k^2}\times\frac{2\mathrm{i}[\sin(kr)-kr\cos(kr)]}{(kr)^2}\,\mathrm{d}k$$

$$= \frac{2P_0\hat{\boldsymbol{r}}}{\pi\varepsilon_0 Rr^2}\int_{0}^{\infty}\frac{[(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)][\sin(kr)-kr\cos(kr)]}{k^4}\,\mathrm{d}k = \begin{cases} -\dfrac{P_0\boldsymbol{r}}{\varepsilon_0 R}; & r<R,\\ -\dfrac{P_0\boldsymbol{r}}{2\varepsilon_0 R}; & r=R,\\ 0; & r>R.\end{cases}$$

The E-field is thus seen to be zero outside the sphere, and given by $\boldsymbol{E}(\boldsymbol{r},t) = -P_0\boldsymbol{r}/(\varepsilon_0 R)$ inside.

Digression: The last integral in part (d) is evaluated using the method of integration-by-parts, as follows:

$$\int_{k=0}^{\infty}\frac{[(kR)^2\sin(kR)+3(kR)\cos(kR)-3\sin(kR)][\sin(kr)-kr\cos(kr)]}{k^4}\,\mathrm{d}k$$

$$= R^3\int_0^{\infty}\frac{(x^2\sin x+3x\cos x-3\sin x)[\sin(rx/R)-(rx/R)\cos(rx/R)]}{x^4}\,\mathrm{d}x$$

$$= R^3\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x - x\cos x}{x^3}\right)[\sin(rx/R)-(rx/R)\cos(rx/R)]\mathrm{d}x$$

$$= R^3\left\{\cancelto{0}{\left(\frac{\sin x - x\cos x}{x^3}\right)[\sin(rx/R)-(rx/R)\cos(rx/R)]\Big|_{x=0}^{\infty}}\right.$$
$$\left. - \int_0^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)(r/R)^2 x\sin(rx/R)\,\mathrm{d}x\right\}$$

$$= Rr^2\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right)\sin(rx/R)\,\mathrm{d}x$$

$$= Rr^2\left\{\cancelto{0}{\frac{\sin x\sin(rx/R)}{x}\Big|_{x=0}^{\infty}} - \int_0^{\infty}(\sin x/x)(r/R)\cos(rx/R)\,\mathrm{d}x\right\}$$

G&R **3.741-2** → $$= -r^3\int_0^{\infty}\frac{\sin x\cos(rx/R)}{x}\,\mathrm{d}x = \begin{cases} -\pi r^3/2; & r<R,\\ -\pi r^3/4; & r=R,\\ 0; & r>R.\end{cases}$$