Opti 501 **2^{nd} Midterm Solutions** (10/26/2017) **Time: 75 minutes**

Problem 1) The principle behind the method of *integration-by-parts* is that, since for any pair of differentiable functions, say, $f(x)$ and $g(x)$, one can write $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$, the definite integral of $f'(x)g(x)$ over the interval $[a, b]$ can be written as

$$\int_a^b f'(x)g(x)\mathrm{d}x = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)\mathrm{d}x = [f(b)g(b) - f(a)g(a)] - \int_a^b f(x)g'(x)\mathrm{d}x. \quad (1)$$

Application of the method of integration-by-parts to the functions $f(x)$ and $\delta'(x - x_0)$ thus yields

$$\int_{-\infty}^{\infty} f(x)\delta'(x - x_0)\mathrm{d}x = f(x)\delta(x - x_0)|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x - x_0)\mathrm{d}x. \quad (2)$$

Now, assuming that $f(x)\delta(x - x_0)$ approaches zero when $x \to \pm\infty$, we can invoke the sifting property of $\delta(x - x_0)$ to simplify Eq.(2), namely,

$$\int_{-\infty}^{\infty} f(x)\delta'(x - x_0)\mathrm{d}x = -f'(x_0). \quad (3)$$

Equation (3) is the general expression of the sifting property of the first derivative $\delta'(x)$ of Dirac's delta-function $\delta(x)$.

Digression: The assumption that $f(x)\delta(x - x_0)$ approaches zero when $x \to \pm\infty$ is obviously valid if $f(x) \to 0$ when $x \to \pm\infty$ (which is usually the case for functions of practical interest), or if the delta-function is defined as $\delta(x) = \lim_{\alpha\to\infty} \alpha g(\alpha x)$, with $g(x)$ being a *finite-width* function of x (that is even and has unit area as well), such as $g(x) = \mathrm{Rect}(x)$ or $g(x) = \mathrm{Tri}(x)$. In such cases, the width of $g(x)$ is a positive real number w, such that $g(x) = 0$ for $|x| > w/2$. The width of $\alpha g(\alpha x)$ will then be w/α and, clearly, $\alpha g[\alpha(x - x_0)]f(x)$ is precisely zero when $x > x_0 + (w/2\alpha)$ or $x < x_0 - (w/2\alpha)$, irrespective of how large the value of the parameter α may be.

One has to be more careful with the assumption $f(x)\delta(x - x_0) \to 0$ when $x \to \pm\infty$, if $f(x)$ fails to approach zero when $x \to \pm\infty$, and the function $g(x)$, chosen to represent $\delta(x)$, happens to have an *infinite extent*, e.g., when $\delta(x) = \lim_{\alpha\to\infty} \alpha\,\mathrm{sinc}(\alpha x)$ or $\delta(x) = \lim_{\alpha\to\infty} \alpha \exp(-\pi\alpha^2 x^2)$. Under such circumstances, one might want to focus attention on a large but finite range for the integration, say, $\int_{-L}^{L} f(x)\delta'(x - x_0)\mathrm{d}x$, with L being a large positive number. The relevant entities appearing on the right-hand-side of Eq.(2) will then be $\alpha g[\alpha(\pm L - x_0)]f(\pm L)$, which must become negligible for sufficiently large α, with the choice of α dictated by the chosen value of L.

Problem 2)

a) $$\rho(\boldsymbol{r}, t) = \rho_0\left[\mathrm{Rect}(x/L_x)\mathrm{Rect}(y/L_y) - \mathrm{Circ}\left(\sqrt{x^2 + y^2}/R\right)\right]\mathrm{Rect}(z/L_z).$$

b) $$\boldsymbol{M}(\boldsymbol{r}, t) = M_0\left[2\mathrm{Rect}(x/L)\mathrm{Rect}(y/L) - \mathrm{Circ}\left(\sqrt{x^2 + y^2}/R\right)\right]\mathrm{Rect}(z/h)\cos(\omega_0 t)\,\hat{\boldsymbol{z}}.$$

Problem 3) a) The current-density distribution may be written as a superposition of plane-waves, as follows:

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{r}, t) &= \tfrac{1}{4}\boldsymbol{J}_1[\exp(\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}) + \exp(-\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r})][\exp(\mathrm{i}\omega_0 t) + \exp(-\mathrm{i}\omega_0 t)] \\ &\quad -\tfrac{1}{4}\boldsymbol{J}_2[\exp(\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}) - \exp(-\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r})][\exp(\mathrm{i}\omega_0 t) - \exp(-\mathrm{i}\omega_0 t)] \\ &= \tfrac{1}{4}(\boldsymbol{J}_1 + \boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t)] + \exp[-\mathrm{i}(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t)]\} \\ &\quad +\tfrac{1}{4}(\boldsymbol{J}_1 - \boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0 \cdot \boldsymbol{r} + \omega_0 t)] + \exp[-\mathrm{i}(\boldsymbol{k}_0 \cdot \boldsymbol{r} + \omega_0 t)]\}. \end{aligned} \quad (1)$$

From the above equation, the Fourier transform of $\boldsymbol{J}(\boldsymbol{r},t)$ is readily seen to be

$$\boldsymbol{J}(\boldsymbol{k},\omega) = \tfrac{1}{4}(2\pi)^4(\boldsymbol{J}_1+\boldsymbol{J}_2)[\delta(\boldsymbol{k}-\boldsymbol{k}_0)\delta(\omega-\omega_0)+\delta(\boldsymbol{k}+\boldsymbol{k}_0)\delta(\omega+\omega_0)]$$
$$+\tfrac{1}{4}(2\pi)^4(\boldsymbol{J}_1-\boldsymbol{J}_2)[\delta(\boldsymbol{k}-\boldsymbol{k}_0)\delta(\omega+\omega_0)+\delta(\boldsymbol{k}+\boldsymbol{k}_0)\delta(\omega-\omega_0)]. \quad (2)$$

b) The charge-current continuity equation, $\boldsymbol{\nabla}\cdot\boldsymbol{J}+\partial\rho/\partial t=0$, yields $\rho(\boldsymbol{r},t)=-\int\boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{r},t)\mathrm{d}t$ and $\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{k},\omega)=\omega\rho(\boldsymbol{k},\omega)$. Consequently,

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{r},t) = \tfrac{1}{4}\mathrm{i}\boldsymbol{k}_0\cdot(\boldsymbol{J}_1+\boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]-\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]\}$$
$$+\tfrac{1}{4}\mathrm{i}\boldsymbol{k}_0\cdot(\boldsymbol{J}_1-\boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]-\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]\}. \quad (3)$$

$$\rho(\boldsymbol{r},t) = \tfrac{1}{4}\omega_0^{-1}\boldsymbol{k}_0\cdot(\boldsymbol{J}_1+\boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]+\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]\}$$
$$-\tfrac{1}{4}\omega_0^{-1}\boldsymbol{k}_0\cdot(\boldsymbol{J}_1-\boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]+\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]\}$$
$$= \tfrac{1}{2}\omega_0^{-1}\boldsymbol{k}_0\cdot[(\boldsymbol{J}_1+\boldsymbol{J}_2)\cos(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)-(\boldsymbol{J}_1-\boldsymbol{J}_2)\cos(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]. \quad (4)$$

$$\rho(\boldsymbol{k},\omega) = \boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{k},\omega)/\omega$$
$$= \tfrac{1}{4}(2\pi)^4\omega_0^{-1}\boldsymbol{k}_0\cdot(\boldsymbol{J}_1+\boldsymbol{J}_2)[\delta(\boldsymbol{k}-\boldsymbol{k}_0)\delta(\omega-\omega_0)+\delta(\boldsymbol{k}+\boldsymbol{k}_0)\delta(\omega+\omega_0)]$$
$$-\tfrac{1}{4}(2\pi)^4\omega_0^{-1}\boldsymbol{k}_0\cdot(\boldsymbol{J}_1-\boldsymbol{J}_2)[\delta(\boldsymbol{k}-\boldsymbol{k}_0)\delta(\omega+\omega_0)+\delta(\boldsymbol{k}+\boldsymbol{k}_0)\delta(\omega-\omega_0)]. \quad (5)$$

c) Given that $k^2-(\omega/c)^2$ is the same for all four Fourier components, all that is needed to find the potentials is dividing the charge and current densities by $k_0^2-(\omega_0/c)^2$. We will have

$$\psi(\boldsymbol{r},t) = \frac{\rho(\boldsymbol{r},t)}{\varepsilon_0[k_0^2-(\omega_0/c)^2]}. \quad (6)$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0\boldsymbol{J}(\boldsymbol{r},t)}{k_0^2-(\omega_0/c)^2}. \quad (7)$$

d) The electric and magnetic fields are now obtained using the standard formulas, as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t)-\frac{\partial\boldsymbol{A}(\boldsymbol{r},t)}{\partial t}$$
$$= \frac{1}{k_0^2-(\omega_0/c)^2}\big(-\tfrac{1}{4}\mathrm{i}\varepsilon_0^{-1}\omega_0^{-1}[(\boldsymbol{J}_1+\boldsymbol{J}_2)\cdot\boldsymbol{k}_0]\boldsymbol{k}_0\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]-\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]\}$$
$$+\tfrac{1}{4}\mathrm{i}\varepsilon_0^{-1}\omega_0^{-1}[(\boldsymbol{J}_1-\boldsymbol{J}_2)\cdot\boldsymbol{k}_0]\boldsymbol{k}_0\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]-\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]\}$$
$$+\tfrac{1}{4}\mathrm{i}\mu_0\omega_0(\boldsymbol{J}_1+\boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]-\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]\}$$
$$-\tfrac{1}{4}\mathrm{i}\mu_0\omega_0(\boldsymbol{J}_1-\boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]-\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]\}\big)$$
$$= \frac{[(\boldsymbol{J}_1+\boldsymbol{J}_2)\cdot\boldsymbol{k}_0\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)-(\boldsymbol{J}_1-\boldsymbol{J}_2)\cdot\boldsymbol{k}_0\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]\boldsymbol{k}_0}{2\varepsilon_0\omega_0[k_0^2-(\omega_0/c)^2]}$$
$$-\frac{\mu_0\omega_0[(\boldsymbol{J}_1+\boldsymbol{J}_2)\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)-(\boldsymbol{J}_1-\boldsymbol{J}_2)\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]}{2[k_0^2-(\omega_0/c)^2]}.$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{\{[(\boldsymbol{J}_1+\boldsymbol{J}_2)\cdot\boldsymbol{k}_0]\boldsymbol{k}_0-(\omega_0/c)^2(\boldsymbol{J}_1+\boldsymbol{J}_2)\}\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)-\{[(\boldsymbol{J}_1-\boldsymbol{J}_2)\cdot\boldsymbol{k}_0]\boldsymbol{k}_0-(\omega_0/c)^2(\boldsymbol{J}_1-\boldsymbol{J}_2)\}\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)}{2\varepsilon_0\omega_0[k_0^2-(\omega_0/c)^2]}. \quad (8)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0\boldsymbol{\nabla}\times\boldsymbol{J}(\boldsymbol{r},t)}{k_0^2-(\omega_0/c)^2}$$

$$= \frac{\mathrm{i}\mu_0\boldsymbol{k}_0}{4[k_0^2-(\omega_0/c)^2]}\times\{(\boldsymbol{J}_1+\boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]-\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)]\}$$

$$+(\boldsymbol{J}_1-\boldsymbol{J}_2)\{\exp[\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]-\exp[-\mathrm{i}(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]\}\}\cdot$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_0[(\boldsymbol{J}_1+\boldsymbol{J}_2)\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t)+(\boldsymbol{J}_1-\boldsymbol{J}_2)\sin(\boldsymbol{k}_0\cdot\boldsymbol{r}+\omega_0 t)]\times\boldsymbol{k}_0}{2[k_0^2-(\omega_0/c)^2]}\,. \qquad (9)$$

It is not difficult to verify that the above expressions for ρ, $\boldsymbol{J}$, $\boldsymbol{E}$, and $\boldsymbol{B}$ satisfy all four of Maxwell's equations.