Problem 1)

a) $\boldsymbol{M}(\boldsymbol{r},t) = M_0\hat{\boldsymbol{z}}\left[\text{Rect}(x/L_x)\text{Rect}(y/L_y) - \text{Circ}(r_\parallel/R)\right]\text{Rect}(z/L_z).$ ← $r_\parallel = \sqrt{x^2+y^2}$

b) $$\rho_{\text{bound}}^{(m)}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r},t) = -\,\partial M_z/\partial z$$
$$= M_0\left[\text{Rect}(x/L_x)\text{Rect}(y/L_y) - \text{Circ}(r_\parallel/R)\right][\delta(z-½L_z) - \delta(z+½L_z)].$$

c) $$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t) = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)$$
$$= \mu_0^{-1}M_0\{\boldsymbol{\nabla}\times\left[\text{Rect}(x/L_x)\text{Rect}(y/L_y)\text{Rect}(z/L_z)\hat{\boldsymbol{z}}\right]$$ ← Cartesian coordinates
$$-\boldsymbol{\nabla}\times[\text{Circ}(r_\parallel/R)\text{Rect}(z/L_z)\hat{\boldsymbol{z}}]\}$$ ← Cylindrical coordinates
$$= \mu_0^{-1}M_0\text{Rect}(x/L_x)[\delta(y+½L_y) - \delta(y-½L_y)]\text{Rect}(z/L_z)\hat{\boldsymbol{x}}$$
$$-\mu_0^{-1}M_0[\delta(x+½L_x) - \delta(x-½L_x)]\text{Rect}(y/L_y)\text{Rect}(z/L_z)\hat{\boldsymbol{y}}$$
$$-\mu_0^{-1}M_0\delta(r_\parallel - R)\text{Rect}(z/L_z)\hat{\boldsymbol{\phi}}.$$

Digression: It is fairly straightforward to evaluate the Fourier transform of the bound charge-density distribution given in part (b). We will have

$$\rho_{\text{bound}}^{(m)}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\rho_{\text{bound}}^{(m)}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t$$
$$= 2\pi M_0\delta(\omega)\{\int_{-\infty}^{\infty}\text{Rect}(x/L_x)\exp(-\mathrm{i}k_x x)\,\mathrm{d}x\int_{-\infty}^{\infty}\text{Rect}(y/L_y)\exp(-\mathrm{i}k_y y)\,\mathrm{d}y$$
$\boldsymbol{k}_\parallel = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$ → $$-\int_{r_\parallel=0}^{\infty}\text{Circ}(r_\parallel/R)\int_{\phi=0}^{2\pi}\exp(-\mathrm{i}k_\parallel r_\parallel\cos\phi)\,r_\parallel\mathrm{d}\phi\mathrm{d}r_\parallel\Big\}$$ ← G&R **3.915**-2
$$\times\int_{-\infty}^{\infty}[\delta(z-½L_z) - \delta(z+½L_z)]\exp(-\mathrm{i}k_z z)\,\mathrm{d}z$$
$$= 2\pi M_0\delta(\omega)\left\{\int_{-L_x/2}^{L_x/2}\exp(-\mathrm{i}k_x x)\,\mathrm{d}x\int_{-L_y/2}^{L_y/2}\exp(-\mathrm{i}k_y y)\,\mathrm{d}y - 2\pi\int_0^R r_\parallel J_0(k_\parallel r_\parallel)\mathrm{d}r_\parallel\right\}$$
$$\times[\exp(-½\mathrm{i}L_z k_z) - \exp(½\mathrm{i}L_z k_z)]$$ ↙ G&R **5.56**-2
$$= 2\pi M_0\delta(\omega)\left\{\frac{2\sin(½L_x k_x)}{k_x}\times\frac{2\sin(½L_y k_y)}{k_y} - \frac{2\pi R}{k_\parallel}J_1(k_\parallel R)\right\}[-2\mathrm{i}\sin(½L_z k_z)]$$
$$= -\mathrm{i}4\pi M_0\delta(\omega)\left\{L_x L_y\text{sinc}\left(\frac{L_x k_x}{2\pi}\right)\text{sinc}\left(\frac{L_y k_y}{2\pi}\right) - 2\pi R\frac{J_1[R(k_x^2+k_y^2)^{½}]}{(k_x^2+k_y^2)^{½}}\right\}\sin(½L_z k_z).$$

Similarly, the Fourier transform of the bound current-density distribution given in part (c), may be computed as follows:

$$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t$$
$$= 2\pi\mu_0^{-1}M_0\delta(\omega)\Big\{\hat{\boldsymbol{x}}\int_{-L_x/2}^{L_x/2}\exp(-\mathrm{i}k_x x)\mathrm{d}x\int_{-\infty}^{\infty}[\delta(y+½L_y) - \delta(y-½L_y)]\exp(-\mathrm{i}k_y y)\mathrm{d}y$$
$$-\hat{\boldsymbol{y}}\int_{-\infty}^{\infty}[\delta(x+½L_x) - \delta(x-½L_x)]\exp(-\mathrm{i}k_x x)\mathrm{d}x\int_{-L_y/2}^{L_y/2}\exp(-\mathrm{i}k_y y)\mathrm{d}y$$
$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}_\parallel$ → $$-\hat{\boldsymbol{z}}\times\int_{r_\parallel=0}^{\infty}\int_{\phi=0}^{2\pi}\hat{\boldsymbol{r}}_\parallel\delta(r_\parallel - R)\exp(-\mathrm{i}k_\parallel r_\parallel\cos\phi)r_\parallel\mathrm{d}r_\parallel\mathrm{d}\phi\Big\}\int_{-L_z/2}^{L_z/2}\exp(-\mathrm{i}k_z z)\mathrm{d}z$$

$$= 2\pi\mu_0^{-1} M_0 \delta(\omega)\{2\mathrm{i}L_x \mathrm{sinc}(L_x k_x/2\pi)\sin(\tfrac{1}{2}L_y k_y)\hat{\boldsymbol{x}} - 2\mathrm{i}L_y \mathrm{sinc}(L_y k_y/2\pi)\sin(\tfrac{1}{2}L_x k_x)\hat{\boldsymbol{y}}$$

G&R **3.915**-2 →
$$-\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_\parallel \int_{r_\parallel=0}^{\infty} r_\parallel \delta(r_\parallel - R)\int_{\phi=0}^{2\pi}\cos\phi \exp(-\mathrm{i}k_\parallel r_\parallel \cos\phi)\mathrm{d}\phi \mathrm{d}r_\parallel\} L_z \mathrm{sinc}(L_z k_z/2\pi)$$

← $\boldsymbol{k}_\parallel \times \hat{\boldsymbol{z}}$

$$= \mathrm{i}2\pi\mu_0^{-1} M_0 \delta(\omega) L_z \mathrm{sinc}(L_z k_z/2\pi)\{L_x L_y \mathrm{sinc}(L_x k_x/2\pi)\,\mathrm{sinc}(L_y k_y/2\pi)(k_y \hat{\boldsymbol{x}} - k_x \hat{\boldsymbol{y}})$$

$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_\parallel$ →
$$+2\pi R\, J_1[R(k_x^2 + k_y^2)^{1/2}]\hat{\boldsymbol{\phi}}\}.$$
← $k_\parallel = (k_x^2 + k_y^2)^{1/2}$

Problem 2)

a) $$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(\boldsymbol{r}) = -\frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} - \frac{\partial\psi}{r\partial\theta}\hat{\boldsymbol{\theta}} = \begin{cases} -(P_0/3\varepsilon_0)\,(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}); & r < R, \\ (P_0 R^3/3\varepsilon_0)\,(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})/r^3; & r \ge R. \end{cases}$$

The field inside the sphere may be further simplified and written as $\boldsymbol{E}(\boldsymbol{r}) = -(P_0/3\varepsilon_0)\hat{\boldsymbol{z}}$.

b) $$\rho_{\text{bound}}^{(e)}(\boldsymbol{r}) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r}) = -\boldsymbol{\nabla}\cdot\left[P_0\,\mathrm{Sphere}(r/R)(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})\right]$$

$$= -\frac{\partial(r^2 P_r)}{r^2\partial r} - \frac{\partial(\sin\theta P_\theta)}{r\sin\theta\partial\theta}$$

$$= -\left(\frac{P_0\cos\theta}{r^2}\right)\frac{\partial[r^2\,\mathrm{Sphere}(r/R)]}{\partial r} + \left[\frac{P_0 \mathrm{Sphere}(r/R)}{r\sin\theta}\right]\frac{\partial\sin^2\theta}{\partial\theta}$$

$$= -\frac{P_0\cos\theta[2r\,\mathrm{Sphere}(r/R) - r^2\,\delta(r-R)]}{r^2} + \frac{2P_0 \mathrm{Sphere}(r/R)\cos\theta}{r} = P_0\delta(r-R)\cos\theta.$$

Because of the δ-function appearing in the above charge-density profile, we can state that the sphere has a bound *surface-charge-density* $\sigma_s(r = R, \theta, \phi) = P_0\cos\theta$. Note that the surface-charge-density is positive on the upper hemisphere and negative on the lower hemisphere.

c) The parallel (or tangential) component of the E-field at the surface of the sphere is E_θ, which is found in (a) to be equal to $(P_0/3\varepsilon_0)\sin\theta$ immediately inside as well as immediately outside the sphere. The continuity requirement for the tangential E-field is, therefore, satisfied.

The perpendicular component of the D-field inside the sphere is given by

$$D_\perp = \varepsilon_0 E_r + P_r = -\tfrac{1}{3}P_0\cos\theta + P_0\cos\theta = \tfrac{2}{3}P_0\cos\theta.$$

Outside the sphere and immediately above the surface, we have $D_\perp = \varepsilon_0 E_r = \tfrac{2}{3}P_0\cos\theta$. Therefore, in the absence of free surface-charge-density, the continuity of $D_\perp$ is confirmed.

If, instead of $D_\perp$, we examine the perpendicular component of the E-field, we find, at the surface of the sphere, the following discontinuity in $E_\perp$:

$$E_r(r = R^+, \theta, \phi) - E_r(r = R^-, \theta, \phi) = (2P_0/3\varepsilon_0)\cos\theta - (-P_0/3\varepsilon_0)\cos\theta$$
$$= (P_0/\varepsilon_0)\cos\theta.$$

This, however, is precisely equal to the bound surface-charge-density $\sigma_s = P_0\cos\theta$ found in part (b), divided by ε_0, which is, once again, consistent with the boundary condition derived from Maxwell's 1st equation, $\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = \varepsilon_0^{-1}\rho_{\text{total}}^{(e)}(\boldsymbol{r},t)$.

Problem 3) a) For an electromagnetic field to be trapped inside a perfectly electrically conducting cavity, the tangential component of the E-field must vanish on all the internal surfaces. Therefore,

$$E_z(r_\parallel = R, \phi, z, t) = E_0\, J_0(\omega_0 R/c)\sin(\omega_0 t) = 0 \quad \rightarrow \quad J_0(\omega_0 R/c) = 0. \tag{1}$$

In words, the cylinder radius must be chosen such that $\omega_0 R/c = 2\pi R/\lambda_0$ is a zero of the Bessel function $J_0(\cdot)$.

b) A surface charge-density exists on the top and bottom facets of the cylindrical can. Invoking Maxwell's first boundary condition, the surface charge-density must equal the perpendicular component of the D-field, namely,

Top and bottom caps: $$\sigma_s(r_\parallel, \phi, z = \pm L/2\,, t) = \mp\varepsilon_0 E_0\, J_0(\omega_0 r_\parallel/c)\sin(\omega_0 t). \tag{2}$$

c) A surface current-density $\boldsymbol{J}_s$ exists wherever the H-field happens to have a component parallel to the surface, that is,

Cylindrical wall: $$\boldsymbol{J}_s(r_\parallel = R, \phi, z, t) = -(E_0/Z_0)\, J_1(\omega_0 R/c)\cos(\omega_0 t)\,\hat{\boldsymbol{z}}. \tag{3}$$

Top and bottom caps: $$\boldsymbol{J}_s(r_\parallel, \phi, z = \pm L/2\,, t) = \pm(E_0/Z_0)\, J_1(\omega_0 r_\parallel/c)\cos(\omega_0 t)\,\hat{\boldsymbol{r}}_\parallel. \tag{4}$$

Note that at $r_\parallel = R$, the current exits the upper cap and enters the cylindrical wall. The opposite happens at the bottom cap, where the current arrives from the cylindrical wall, then enters the cap from the rim located at $r_\parallel = R$.

d) On the cylindrical facet, the charge-density is zero, and so is the divergence of the surface current-density $\boldsymbol{J}_s$ given by Eq.(3). This confirms that, on the interior cylindrical wall, the continuity equation is satisfied.

At the top and bottom caps we have

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{J}_s(r_\parallel, \phi, z = \pm L/2\,, t) &= \pm(E_0/Z_0)\frac{\partial[r_\parallel J_1(\omega_0 r_\parallel/c)]}{r_\parallel \partial r_\parallel}\cos(\omega_0 t)\\
&= \pm(E_0/Z_0)\frac{J_1(\omega_0 r_\parallel/c) + (\omega_0 r_\parallel/c)J_1'(\omega_0 r_\parallel/c)}{r_\parallel}\cos(\omega_0 t)\\
&= \pm(E_0/Z_0)\frac{(\omega_0 r_\parallel/c)J_0(\omega_0 r_\parallel/c)}{r_\parallel}\cos(\omega_0 t)\\
&= \pm\varepsilon_0 E_0 \omega_0\, J_0(\omega_0 r_\parallel/c)\cos(\omega_0 t).
\end{aligned} \tag{5}$$

Clearly, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_s + \partial\sigma_s/\partial t = 0$; see Eqs.(2) and (5).

Digression: Below we confirm that the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields given in the statement of the problem do in fact satisfy Maxwell's equations.

1) $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \varepsilon_0\, \partial E_z/\partial z = 0$

2) $\boldsymbol{\nabla}\times\boldsymbol{H} = \frac{\partial(r_\parallel H_\phi)}{r_\parallel \partial r_\parallel}\hat{\boldsymbol{z}} = (E_0/Z_0)\ \frac{\partial[r_\parallel J_1(\omega_0 r_\parallel/c)]}{r_\parallel \partial r_\parallel}\cos(\omega_0 t)\,\hat{\boldsymbol{z}}$

$$= (E_0/Z_0)\frac{J_1(\omega_0 r_\parallel/c) + (\omega_0 r_\parallel/c)J_1'(\omega_0 r_\parallel/c)}{r_\parallel}\cos(\omega_0 t)\,\hat{\boldsymbol{z}}$$

$$= (E_0/Z_0)\frac{(\omega_0 r_\parallel/c)J_0(\omega_0 r_\parallel/c)}{r_\parallel}\cos(\omega_0 t)\,\hat{\boldsymbol{z}}$$

$$= \varepsilon_0 E_0 \omega_0\, J_0(\omega_0 r_\parallel/c)\cos(\omega_0 t)\,\hat{\boldsymbol{z}} = \varepsilon_0\, \partial \boldsymbol{E}/\partial t$$

3) $$\boldsymbol{\nabla}\times\boldsymbol{E} = -(\partial E_z/\partial r_\parallel)\hat{\boldsymbol{\phi}} = -E_0(\omega_0/c)\, J_0'(\omega_0 r_\parallel/c)\sin(\omega_0 t)\,\hat{\boldsymbol{\phi}}$$ ← $J_0'(x) = -J_1(x)$ Chap.3, Eq.(39)

$$= E_0(\omega_0/c)\, J_1(\omega_0 r_\parallel/c)\sin(\omega_0 t)\,\hat{\boldsymbol{\phi}} = -\mu_0\, \partial \boldsymbol{H}/\partial t.$$

4) $$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \mu_0 r_\parallel^{-1}\, \partial H_\phi/\partial\phi = 0.$$
