Problem 1)

a) The current-density $J_{s1}\hat{\boldsymbol{z}}$ of the inner cylinder produces a current-density $(R_1/\rho)J_{s1}\hat{\boldsymbol{\rho}}$ in the upper end-cap as the current leaves the inner cylinder and moves radially outward toward the outer cylinder; here ρ is the radial distance from the cylinder axis. The current-density in the outer cylinder is then given by $\boldsymbol{J}_{s2} = -(R_1/R_2)J_{s1}\hat{\boldsymbol{z}}$. The current returns to the inner cylinder via the lower end-cap, where the current-density $-(R_1/\rho)J_{s1}\hat{\boldsymbol{\rho}}$ is equal in magnitude but opposite in direction to that in the upper cap.

b) This is a magneto-static problem involving time-independent current and no charges; therefore, there is no E-field and the magnetic field throughout the entire space is going to be time-independent. Based on our knowledge of infinitely-long cylinders with a uniform current flowing along their axis of symmetry, we suspect the magnetic field in the present problem to be azimuthally directed within the cavity (i.e., in the region between the two cylinders), with a magnitude that drops in proportion to the inverse of the distance ρ from the cylinder axis, that is, $\boldsymbol{H}(\boldsymbol{r}) = H_0\hat{\boldsymbol{\varphi}}/\rho$. The unknown constant H_0 is determined by matching the boundary conditions. The H-field inside the inner cylinder (i.e., in the region $\rho < R_1$) is expected to be zero.

Now, at the surface of the inner cylinder, the discontinuity in the H-field along $\hat{\boldsymbol{\varphi}}$ will be H_0/R_1, which must be equal to the surface current-density J_{s1} along $\hat{\boldsymbol{z}}$. Consequently, $H_0 = R_1 J_{s1}$. The magnetic field trapped in the region between the two cylinders is thus seen to be

$$\boldsymbol{H}(\boldsymbol{r},t) = \left(\frac{R_1 J_{s1}}{\rho}\right)\hat{\boldsymbol{\varphi}}; \qquad R_1 < \rho < R_2, \ \ |z| < \tfrac{1}{2}L. \tag{1}$$

The magnitude of the above field is equal to the surface current-density of the outer cylinder at $\rho = R_2$, and also equal to the surface current-densities of the end-caps at $z = \pm\tfrac{1}{2}L$. The orientation of the above H-field at the inner walls of the cavity is also consistent with Maxwell's boundary condition at these surfaces. It is thus seen that the H-field outside the cavity must vanish everywhere for the boundary condition (i.e., H-field discontinuity = surface current-density) to be satisfied at all four surfaces. We conclude that the magnetic field outside the cavity is zero everywhere.

Maxwell's relevant equations for magnetostatics are $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}_{\text{free}}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r}) = 0$. In the present problem, $\boldsymbol{B}(\boldsymbol{r}) = \mu_0\boldsymbol{H}(\boldsymbol{r})$ everywhere. Outside the cavity, the H-field is zero, which obviously satisfies both equations. Inside the cavity, $\boldsymbol{J}_{\text{free}} = 0$; therefore, both $\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r})$ and $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r})$ must vanish. We have

$$\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r}) = \frac{\partial(\rho H_\rho)}{\rho\partial\rho} + \frac{\partial H_\varphi}{\rho\partial\varphi} + \frac{\partial H_z}{\partial z} = \frac{\partial H_\varphi}{\rho\partial\varphi} = 0. \tag{2}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) &= \left(\frac{\partial H_z}{\rho\partial\varphi} - \frac{\partial H_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial\rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left[\frac{\partial(\rho H_\varphi)}{\partial\rho} - \frac{\partial H_\rho}{\partial\varphi}\right]\hat{\boldsymbol{z}} \\ &= -\frac{\partial H_\varphi}{\partial z}\hat{\boldsymbol{\rho}} + \frac{\partial(\rho H_\varphi)}{\rho\partial\rho}\hat{\boldsymbol{z}} = 0. \end{aligned} \tag{3}$$

Thus the magnetic field of Eq.(1) satisfies all the relevant equations of Maxwell. In addition, when the H-field outside the cavity is assumed to be zero, the boundary conditions at the inner walls of the cavity are simultaneously satisfied. The uniqueness of solutions of Maxwell's equations thus guarantees that the correct field distribution has been identified.

Problem 2)

a) $$\boldsymbol{A}(\boldsymbol{r},t) = A_0\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\sin\theta\cos(\omega t)\,\hat{\boldsymbol{\varphi}}.$$

Using the Taylor series expansion of sine and cosine functions, we write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

Therefore, in the limit when $r \to 0$, we have

$$\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r} = \left[\frac{1}{k_0 r} - \frac{k_0 r}{3!} + \frac{(k_0 r)^3}{5!} - \cdots\right] - \left[\frac{1}{k_0 r} - \frac{k_0 r}{2!} + \frac{(k_0 r)^3}{4!} - \cdots\right] = \frac{k_0 r}{3} - \frac{(k_0 r)^3}{30} + \cdots.$$

It is thus seen that $A_\varphi(\boldsymbol{r},t)$ approaches zero when $r \to 0$, and that, therefore, the vector potential does not have a singularity at the origin.

b) In the Lorenz gauge, $\boldsymbol{\nabla}\cdot\boldsymbol{A} + (1/c^2)\,\partial\psi/\partial t = 0$. In the present problem, since $\psi(\boldsymbol{r},t) = 0$, it is sufficient to show that $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. Considering that the only component of $\boldsymbol{A}(\boldsymbol{r},t)$ is A_φ, which is independent of the azimuthal angle φ, we have $\boldsymbol{\nabla}\cdot\boldsymbol{A} = (r\sin\theta)^{-1}\partial A_\varphi/\partial\varphi = 0$. The Lorenz gauge requirement is therefore satisfied.

c) $$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi - \frac{\partial\boldsymbol{A}}{\partial t} = A_0\omega\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\sin\theta\sin(\omega t)\,\hat{\boldsymbol{\varphi}}.$$

d) $$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\varphi)}{\partial\theta}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}\hat{\boldsymbol{\theta}}$$
$$= \frac{A_0}{r}\left\{\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) - \sin(k_0 r)\sin\theta\,\hat{\boldsymbol{\theta}}\right\}\cos(\omega t).$$

Note that $\mathcal{L}im_{\,r\to 0}\,\boldsymbol{B}(\boldsymbol{r},t) = ⅔k_0A_0\left(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right)\cos(\omega t) = ⅔k_0A_0\hat{\boldsymbol{z}}\cos(\omega t)$ is regular.

e) $$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}\times\boldsymbol{H} = \frac{(A_0\omega)^2}{2Z_0k_0r}\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\left\{\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\left(2\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\theta\,\hat{\boldsymbol{r}}\right)\right.$$
$$\left. + \sin(k_0 r)\sin\theta\,\hat{\boldsymbol{r}}\right\}\sin\theta\sin(2\omega t).$$

Since the time-averaged $\boldsymbol{S}(\boldsymbol{r},t)$ is zero, the electromagnetic energy is essentially stationary.

Problem 3)

a) $$\rho(\boldsymbol{r},t) = \lambda_0\delta(x)\delta(y)\mathrm{Rect}\left(\frac{z}{2L}\right).$$

b) $$\psi(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\rho(\boldsymbol{r}',t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}' = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda_0\delta(x')\delta(y')\mathrm{Rect}(z'/2L)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dx'dy'dz'$$
$$= \frac{\lambda_0}{4\pi\varepsilon_0}\int_{z'=-L}^{L}\frac{dz'}{\sqrt{x^2+y^2+(z'-z)^2}} = \frac{\lambda_0}{4\pi\varepsilon_0}\ln\left[(z'-z) + \sqrt{x^2+y^2+(z'-z)^2}\right]\Big|_{z'=-L}^{L}$$
$$= \frac{\lambda_0}{4\pi\varepsilon_0}\ln\left[\frac{\sqrt{r^2+(L-z)^2}+(L-z)}{\sqrt{r^2+(L+z)^2}-(L+z)}\right] = \frac{\lambda_0}{4\pi\varepsilon_0}\ln\left\{\frac{\left[\sqrt{r^2+(L-z)^2}+(L-z)\right]\left[\sqrt{r^2+(L+z)^2}+(L+z)\right]}{r^2}\right\}.$$

c) Introducing the normalized parameters $\tilde{r} = r/L$ and $\tilde{z} = z/L$, the above equation may be written as follows:

$$\psi(\boldsymbol{r},t) = -\frac{\lambda_0 \ln r}{2\pi\varepsilon_0} + \frac{\lambda_0 \ln L}{2\pi\varepsilon_0} + \frac{\lambda_0}{4\pi\varepsilon_0}\ln\left\{\left[\sqrt{(1-\tilde{z})^2+\tilde{r}^2}+(1-\tilde{z})\right]\left[\sqrt{(1+\tilde{z})^2+\tilde{r}^2}+(1+\tilde{z})\right]\right\}.$$

In the limit when $L \to \infty$, both $\tilde{r}$ and $\tilde{z}$ approach zero, and the above equation becomes

$$\psi(\boldsymbol{r},t) = \frac{\lambda_0 \ln(2L)}{2\pi\varepsilon_0} - \frac{\lambda_0 \ln r}{2\pi\varepsilon_0}.$$

The large constant containing $\ln(2L)$ in the above expression does *not* contribute to the gradient of the scalar potential. Therefore, the E-field of the infinitely-long rod is given by

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi = -\left(\frac{\partial\psi}{\partial r}\right)\hat{\boldsymbol{r}} = \frac{\lambda_0}{2\pi\varepsilon_0 r}\hat{\boldsymbol{r}}.$$

d) The Fourier transform of the charge-density distribution is given by

$$\begin{aligned}\rho(\boldsymbol{k},\omega) &= \int_{-\infty}^{\infty}\rho(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{r}dt\\ &= 2\pi\delta(\omega)\lambda_0\int_{-L}^{L}\exp(-\mathrm{i}k_z z)\,dz = 4\pi\lambda_0\delta(\omega)\sin(Lk_z)/k_z.\end{aligned}$$

Since the Fourier-transformed scalar potential is $\psi(\boldsymbol{k},\omega) = \varepsilon_0^{-1}\rho(\boldsymbol{k},\omega)/[k^2-(\omega/c)^2]$, its inverse transform may now be evaluated as follows:

$$\begin{aligned}\psi(\boldsymbol{r},t) &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\psi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{k}d\omega\\
&= \frac{2\lambda_0}{(2\pi)^3\varepsilon_0}\int_{-\infty}^{\infty}\frac{\sin(Lk_z)}{k_z\,k^2}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{k}\\
&= \frac{2\lambda_0}{(2\pi)^3\varepsilon_0}\int_{-\infty}^{\infty}\frac{\sin(Lk_z)\exp(\mathrm{i}k_z z)}{k_z(k_x^2+k_y^2+k_z^2)}\exp\left[\mathrm{i}\left(k_x x+k_y y\right)\right]dk_x dk_y dk_z \longleftarrow \boxed{\text{Define } \boldsymbol{k}_\parallel = k_x\hat{\boldsymbol{x}}+k_y\hat{\boldsymbol{y}} \text{ and } \boldsymbol{r}_\parallel = x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}.}\\
&= \frac{2\lambda_0}{(2\pi)^3\varepsilon_0}\int_{k_z=-\infty}^{\infty}\frac{\sin(Lk_z)[\cos(k_z z)+\mathrm{i}\sin(k_z z)]}{k_z}\int_{k_\parallel=0}^{\infty}\frac{1}{k_\parallel^2+k_z^2}\int_{\varphi=0}^{2\pi}\exp(\mathrm{i}k_\parallel r_\parallel\cos\varphi)k_\parallel d\varphi\,dk_\parallel dk_z\\
&= \frac{\lambda_0}{(2\pi)^2\varepsilon_0}\int_{-\infty}^{\infty}\frac{\{\sin[k_z(L+z)]+\sin[k_z(L-z)]\}+\mathrm{i}\{\cos[k_z(L-z)]-\cos[k_z(L+z)]\}}{k_z}\int_{k_\parallel=0}^{\infty}\frac{k_\parallel J_0(k_\parallel r_\parallel)}{k_\parallel^2+k_z^2}dk_\parallel dk_z\\
&= \frac{\lambda_0}{(2\pi)^2\varepsilon_0}\int_{-\infty}^{\infty}\frac{\{\sin[k_z(L+z)]+\sin[k_z(L-z)]\}-\mathrm{i}\{\cos[k_z(L+z)]-\cos[k_z(L-z)]\}}{k_z}K_0(r_\parallel|k_z|)dk_z\\
&= \frac{\lambda_0}{(2\pi)^2\varepsilon_0}\int_{-\infty}^{\infty}k_z^{-1}\{\sin[k_z(L+z)]+\sin[k_z(L-z)]\}K_0(r_\parallel|k_z|)dk_z\\
&= \frac{\pi\lambda_0}{(2\pi)^2\varepsilon_0}\left\{\ln\left[\left(\frac{L+z}{r_\parallel}\right)+\sqrt{1+\left(\frac{L+z}{r_\parallel}\right)^2}\right]+\ln\left[\left(\frac{L-z}{r_\parallel}\right)+\sqrt{1+\left(\frac{L-z}{r_\parallel}\right)^2}\right]\right\}\\
&= -\frac{\lambda_0\ln r_\parallel}{2\pi\varepsilon_0}+\frac{\lambda_0}{4\pi\varepsilon_0}\ln\left\{\left[(L+z)+\sqrt{r_\parallel^2+(L+z)^2}\right]\left[(L-z)+\sqrt{r_\parallel^2+(L-z)^2}\right]\right\}.\end{aligned}$$

The terms of the integrand that contain cosines are odd functions of k_z; therefore, their integrals vanish.

This result is identical with that obtained in part (b), which was obtained using direct evaluation in the spacetime domain.