Problem 1)

a) Charge-density distribution: $\rho(\boldsymbol{r},t)=\sigma_{s0}\mathrm{Circ}(r_\parallel/R)\delta(z)$, where $r_\parallel=\sqrt{x^2+y^2}$.

b) Polarization distribution: $\boldsymbol{P}(\boldsymbol{r},t)=P_0\hat{\boldsymbol{x}}\,\mathrm{Rect}(x/L_x)\mathrm{Rect}(y/L_y)\mathrm{Rect}(z/L_z)$.

c) Magnetization distribution: $\boldsymbol{M}(\boldsymbol{r},t)=M_0\hat{\boldsymbol{z}}\,\mathrm{Circ}(r_\parallel/R)\mathrm{Rect}(z/h)\cos(\omega_0 t+\varphi_0)$.

Problem 2) a) For the incident plane-wave in the region $y\le 0$, we have

$$\boldsymbol{E}^{(\mathrm{inc})}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\psi-\partial\boldsymbol{A}/\partial t=\omega_0A_0\hat{\boldsymbol{z}}\cos(k_0y-\omega_0t),$$

$$\boldsymbol{H}^{(\mathrm{inc})}(\boldsymbol{r},t)=\mu_0^{-1}\boldsymbol{B}(\boldsymbol{r},t)=\mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t)=\mu_0^{-1}(\partial A_z/\partial y)\hat{\boldsymbol{x}}$$
$$=\mu_0^{-1}k_0A_0\hat{\boldsymbol{x}}\cos(k_0y-\omega_0t)=(\omega_0A_0/Z_0)\hat{\boldsymbol{x}}\cos(k_0y-\omega_0t).$$

For the reflected plane-wave (again in the region $y\le 0$), we have

$$\boldsymbol{E}^{(\mathrm{ref})}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\psi-\partial\boldsymbol{A}/\partial t=-\omega_0A_0\hat{\boldsymbol{z}}\cos(k_0y+\omega_0t),$$

$$\boldsymbol{H}^{(\mathrm{ref})}(\boldsymbol{r},t)=\mu_0^{-1}\boldsymbol{B}(\boldsymbol{r},t)=\mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t)=\mu_0^{-1}(\partial A_z/\partial y)\hat{\boldsymbol{x}}$$
$$=\mu_0^{-1}k_0A_0\hat{\boldsymbol{x}}\cos(k_0y+\omega_0t)=(\omega_0A_0/Z_0)\hat{\boldsymbol{x}}\cos(k_0y+\omega_0t).$$

b) In the plane $y=0$ at the front facet of the mirror, the total E-field and the total H-field are given by

$$\boldsymbol{E}^{(\mathrm{total})}(x,y=0,z,t)=\boldsymbol{E}^{(\mathrm{inc})}+\boldsymbol{E}^{(\mathrm{ref})}=\omega_0A_0\hat{\boldsymbol{z}}\cos(-\omega_0t)-\omega_0A_0\hat{\boldsymbol{z}}\cos(\omega_0t)=0,$$

$$\boldsymbol{H}^{(\mathrm{total})}(x,y=0,z,t)=\boldsymbol{H}^{(\mathrm{inc})}+\boldsymbol{H}^{(\mathrm{ref})}=2(\omega_0A_0/Z_0)\hat{\boldsymbol{x}}\cos(\omega_0t).$$

There is no perpendicular E-field immediately before the mirror at $y=0^-$. Also, inside the mirror, and specifically at $y=0^+$, there are no E-fields. Maxwell's boundary condition relating the surface charge-density to the discontinuity of $\varepsilon_0\boldsymbol{E}_\perp$ at $y=0$ thus yields $\sigma_s(x,z,t)=0$.

The tangential H-field immediately before the mirror at $y=0^-$ is $2(\omega_0A_0/Z_0)\hat{\boldsymbol{x}}\cos(\omega_0t)$. Since inside the mirror, and specifically at $y=0^+$, there exist no H-fields, Maxwell's boundary condition relating the surface current-density $\boldsymbol{J}_s$ to the discontinuity of $\boldsymbol{H}_\parallel$ at $y=0$ yields $\boldsymbol{J}_s(x,z,t)=2(\omega_0A_0/Z_0)\hat{\boldsymbol{z}}\cos(\omega_0t)$. The amplitude of this surface current-density is thus given by $J_{s0}=2(\omega_0A_0/Z_0)$.

c) According to Example 10, Chapter 4, the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of the plane-wave propagating in the region $y\ge 0$ are $\boldsymbol{E}(\boldsymbol{r},t)=-½Z_0J_{s0}\hat{\boldsymbol{z}}\cos(k_0y-\omega_0t)$ and $\boldsymbol{H}(\boldsymbol{r},t)=-½J_{s0}\hat{\boldsymbol{x}}\cos(k_0y-\omega_0t)$. We may also consider the vector potential of the field radiated into the shadow region, which is given by $\boldsymbol{A}(\boldsymbol{r},t)=-½(Z_0J_{s0}/\omega_0)\hat{\boldsymbol{z}}\sin(k_0y-\omega_0t)$. Clearly, the field radiated into the shadow region is exactly cancelled out by the continuation beyond the PEC mirror of the incident beam.

Problem 3)

a) For the plane-wave, the Lorenz gauge formula $\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{\partial\psi}{c^2\partial t}=0$ becomes $\boldsymbol{k}\cdot\boldsymbol{A}_0=(\omega/c^2)\psi_0$.

b)
$$\boldsymbol{E}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\psi-\partial\boldsymbol{A}/\partial t=(-\mathrm{i}\boldsymbol{k}\psi_0+\mathrm{i}\omega\boldsymbol{A}_0)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)],$$
$$\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{\nabla}\times\boldsymbol{A}=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].$$

c) i) Maxwell's first equation (in free space): $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0 \;\rightarrow\; \boldsymbol{k}\cdot\boldsymbol{E}=0 \;\rightarrow\; k^2\psi_0-\omega\boldsymbol{k}\cdot\boldsymbol{A}_0=0$. This result may now be combined with that obtained in part (a) to yield

$$[k^2-(\omega/c)^2]\psi_0=0.$$

If $\psi_0\neq 0$, we must have $k^2=(\omega/c)^2$.

ii) Maxwell's second equation (in free space):

$$\boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\varepsilon_0\,\partial\boldsymbol{E}/\partial t \quad\rightarrow\quad \mathrm{i}^2\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{A}_0)=-\mathrm{i}(\omega/c^2)(-\mathrm{i}\boldsymbol{k}\psi_0+\mathrm{i}\omega\boldsymbol{A}_0)$$
$$\rightarrow\quad (\boldsymbol{k}\cdot\boldsymbol{A}_0)\boldsymbol{k}-k^2\boldsymbol{A}_0=(\omega/c^2)(\boldsymbol{k}\psi_0-\omega\boldsymbol{A}_0)$$
$$\rightarrow\quad [\boldsymbol{k}\cdot\boldsymbol{A}_0-(\omega/c^2)\psi_0]\boldsymbol{k}=[k^2-(\omega/c)^2]\boldsymbol{A}_0$$

The preceding equation, when combined with the Lorenz gauge result obtained in part (a), yields

$$[k^2-(\omega/c)^2]\boldsymbol{A}_0=0,$$

which gives a non-zero value for $\boldsymbol{A}_0$ only if $k^2=(\omega/c)^2$.

iii) Maxwell's third equation:

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\,\partial\boldsymbol{B}/\partial t \quad\rightarrow\quad \mathrm{i}\boldsymbol{k}\times(-\mathrm{i}\boldsymbol{k}\psi_0+\mathrm{i}\omega\boldsymbol{A}_0)=\mathrm{i}^2\omega\boldsymbol{k}\times\boldsymbol{A}_0$$
$$\rightarrow\quad -\underbrace{(\boldsymbol{k}\times\boldsymbol{k})}_{0}\psi_0+\omega\boldsymbol{k}\times\boldsymbol{A}_0=\omega\boldsymbol{k}\times\boldsymbol{A}_0 \quad \text{(automatically satisfied).}$$

iv) Maxwell's fourth equation:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \quad\rightarrow\quad \mathrm{i}^2\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{A}_0)=0 \quad\rightarrow\quad \overbrace{(\boldsymbol{k}\times\boldsymbol{k})}^{0}\cdot\boldsymbol{A}_0=0 \quad \text{(automatically satisfied).}$$

Problem 4) a) The charge-current continuity equation yields

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}=\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}=-\frac{\partial\rho}{\partial t}=\omega_0\rho_0\cos(k_0x)\sin(\omega_0 t).$$

It is sufficient to assume that $J_y=J_z=0$, and that J_x is a function only of x and t. We will have

$$\frac{\partial}{\partial x}J_x(x,t)=\omega_0\rho_0\cos(k_0x)\sin(\omega_0t) \quad\rightarrow\quad \boldsymbol{J}(\boldsymbol{r},t)=J_x\hat{\boldsymbol{x}}=\left(\frac{\omega_0\rho_0}{k_0}\right)\sin(k_0x)\sin(\omega_0t)\,\hat{\boldsymbol{x}}.$$

b) $\rho(\boldsymbol{r},t)$ and $\boldsymbol{J}(\boldsymbol{r},t)$ may be expressed as superpositions of plane-waves, as follows:

$$\rho(\boldsymbol{r},t)=\tfrac{1}{4}\rho_0[\exp(\mathrm{i}k_0x)+\exp(-\mathrm{i}k_0x)][\exp(\mathrm{i}\omega_0t)+\exp(-\mathrm{i}\omega_0t)]$$
$$=\tfrac{1}{4}\rho_0\exp[\mathrm{i}(k_0x+\omega_0t)]+\tfrac{1}{4}\rho_0\exp[-\mathrm{i}(k_0x+\omega_0t)]$$
$$+\tfrac{1}{4}\rho_0\exp[\mathrm{i}(k_0x-\omega_0t)]+\tfrac{1}{4}\rho_0\exp[-\mathrm{i}(k_0x-\omega_0t)].$$

$$\boldsymbol{J}(\boldsymbol{r},t)=-\left(\frac{\omega_0\rho_0}{4k_0}\right)[\exp(\mathrm{i}k_0x)-\exp(-\mathrm{i}k_0x)][\exp(\mathrm{i}\omega_0t)-\exp(-\mathrm{i}\omega_0t)]\hat{\boldsymbol{x}}$$
$$=-\left(\frac{\omega_0\rho_0}{4k_0}\right)\exp[\mathrm{i}(k_0x+\omega_0t)]\,\hat{\boldsymbol{x}}-\left(\frac{\omega_0\rho_0}{4k_0}\right)\exp[-\mathrm{i}(k_0x+\omega_0t)]\,\hat{\boldsymbol{x}}$$
$$+\left(\frac{\omega_0\rho_0}{4k_0}\right)\exp[\mathrm{i}(k_0x-\omega_0t)]\,\hat{\boldsymbol{x}}+\left(\frac{\omega_0\rho_0}{4k_0}\right)\exp[-\mathrm{i}(k_0x-\omega_0t)]\,\hat{\boldsymbol{x}}.$$

All the above plane-waves have $k_x=\pm k_0$, $k_y=k_z=0$, and $\omega=\pm\omega_0$. Therefore, the value of $[k^2-(\omega/c)^2]$ for all these plane-waves is the same, namely, $[k_0^2-(\omega_0/c)^2]$. The scalar potential for each charge-density plane-wave is obtained by multiplying the corresponding charge-density by $\frac{1}{\varepsilon_0[k_0^2-(\omega_0/c)^2]}$. Similarly, the vector potential for each current-density plane-wave is obtained by multiplying the corresponding current-density by $\frac{\mu_0}{k_0^2-(\omega_0/c)^2}$. It is readily observed that the scalar potential and vector potential plane-waves combine once again to form simple sine and cosine functions, as follows:

$$\psi(\boldsymbol{r},t) = \frac{\rho_0 \cos(k_0 x)\cos(\omega_0 t)}{\varepsilon_0[k_0^2-(\omega_0/c)^2]},$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{(\mu_0\omega_0\rho_0/k_0)\sin(k_0 x)\sin(\omega_0 t)}{k_0^2-(\omega_0/c)^2}\hat{\boldsymbol{x}}.$$

c) $$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$= \frac{\rho_0 k_0 \sin(k_0 x)\cos(\omega_0 t)}{\varepsilon_0[k_0^2-(\omega_0/c)^2]}\hat{\boldsymbol{x}} - \frac{(\mu_0\omega_0^2\rho_0/k_0)\sin(k_0 x)\cos(\omega_0 t)}{k_0^2-(\omega_0/c)^2}\hat{\boldsymbol{x}}$$

$$= \left(\frac{\rho_0}{\varepsilon_0 k_0}\right)\sin(k_0 x)\cos(\omega_0 t)\,\hat{\boldsymbol{x}}.$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A} = (\partial A_x/\partial z)\hat{\boldsymbol{y}} - (\partial A_x/\partial y)\hat{\boldsymbol{z}} = 0.$$

It is interesting to note that neither the current $\boldsymbol{J}(\boldsymbol{r},t)$ nor the time-dependent E-field in the present problem give rise to a magnetic field. In fact, a quick check of Maxwell's second equation reveals that $\boldsymbol{J}(\boldsymbol{r},t)$ and $\partial\boldsymbol{D}/\partial t$ exactly cancel out. The satisfaction of the remaining Maxwell's equations may also be readily verified.