Problem 1) The function $f(t) = \cos(\omega_0 t)$ must first be multiplied by $\exp(-\alpha|t|)$ in order to prevent its Fourier integral from diverging as $t \to \pm\infty$. Eventually, however, we must let $\alpha \to 0$, to lift this artificial restriction on the magnitude of $f(t)$.

$$
\begin{aligned}
F(\omega) &= \lim_{\alpha\to 0} \int_{-\infty}^{\infty} \cos(\omega_0 t)\exp(-\alpha|t|)\exp(i\omega t)\,dt \\
&= \lim_{\alpha\to 0} \int_{-\infty}^{\infty} ½[\exp(i\omega_0 t) + \exp(-i\omega_0 t)]\exp(-\alpha|t|)\exp(i\omega t)\,dt \\
&= ½\lim_{\alpha\to 0} \Big\{\int_{-\infty}^{0}[\exp(i\omega_0 t) + \exp(-i\omega_0 t)]\exp(\alpha t)\exp(i\omega t)\,dt \\
&\qquad + \int_{0}^{\infty}[\exp(i\omega_0 t) + \exp(-i\omega_0 t)]\exp(-\alpha t)\exp(i\omega t)\,dt\Big\} \\
&= ½\lim_{\alpha\to 0} \Big\{\int_{-\infty}^{0}\exp\{[\alpha + i(\omega+\omega_0)]t\}dt + \int_{-\infty}^{0}\exp\{[\alpha + i(\omega-\omega_0)]t\}\,dt \\
&\qquad + \int_{0}^{\infty}\exp\{-[\alpha - i(\omega+\omega_0)]t\}\,dt + \int_{0}^{\infty}\exp\{-[\alpha - i(\omega-\omega_0)]t\}\,dt\Big\} \\
&= \lim_{\alpha\to 0}\left[\frac{½}{\alpha+i(\omega+\omega_0)} + \frac{½}{\alpha+i(\omega-\omega_0)} + \frac{½}{\alpha-i(\omega+\omega_0)} + \frac{½}{\alpha-i(\omega-\omega_0)}\right] \\
&= \lim_{\alpha\to 0}\left[\frac{½}{\alpha+i(\omega+\omega_0)} + \frac{½}{\alpha-i(\omega+\omega_0)} + \frac{½}{\alpha+i(\omega-\omega_0)} + \frac{½}{\alpha-i(\omega-\omega_0)}\right] \\
&= \lim_{\alpha\to 0}\left[\frac{\alpha}{\alpha^2+(\omega+\omega_0)^2} + \frac{\alpha}{\alpha^2+(\omega-\omega_0)^2}\right].
\end{aligned}
$$

The area under each of the two functions in the preceding expression is equal to π, as may be readily verified:

$$
\int_{-\infty}^{\infty}\frac{\alpha}{\alpha^2+(\omega\pm\omega_0)^2}\,d\omega = \int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-½\pi}^{½\pi}\left(\frac{1+\tan^2\theta}{1+\tan^2\theta}\right)d\theta = \pi.
$$

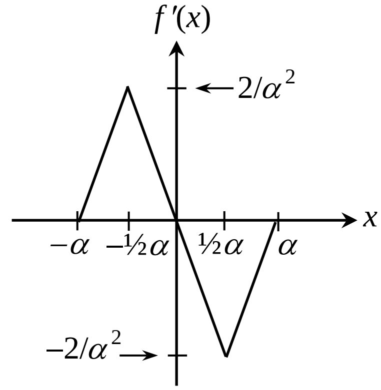
In the limit when $\alpha \to 0$, the first of the above functions approaches $\pi\delta(\omega+\omega_0)$ while the second one approaches $\pi\delta(\omega-\omega_0)$. Consequently, $F(\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$.

Problem 2) a) The symmetry of the function $f(x)$ allows us to integrate over the interval $(0,\alpha)$, then multiply the result by 2 to find the area under the function, that is,

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= 2\left\{\int_{0}^{½\alpha}(2/\alpha)[½ - (x/\alpha)^2]dx + \int_{½\alpha}^{\alpha}(2/\alpha)[1-(x/\alpha)]^2dx\right\} \\
&= 4\left[\int_{0}^{½}(½ - y^2)dy + \int_{½}^{1}(1-y)^2dy\right] \\
&= 4\left[(½y - ⅓y^3)\big|_0^{½} - ⅓(1-y)^3\big|_{½}^{1}\right] = 4\left(\frac{1}{4} - \frac{1}{24} + \frac{1}{24}\right) = 1
\end{aligned}
$$

Since $f(x)$ is tall, narrow, symmetric around $x = 0$, and has unit area, in the limit when $\alpha \to 0$, the function $f(x)$ approaches a delta-function.

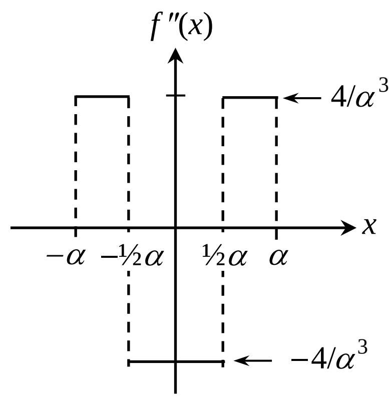
b) Differentiation of $f(x)$ with respect to x yields



$$f'(x) = \begin{cases} 0, & x \le -\alpha; \\ (4/\alpha^2)[1 + (x/\alpha)], & -\alpha \le x \le -½\alpha; \\ -4x/\alpha^3, & -½\alpha \le x \le ½\alpha; \\ -(4/\alpha^2)[1 - (x/\alpha)], & ½\alpha \le x \le \alpha; \\ 0, & x \ge \alpha. \end{cases}$$

The area under each half of the function $f'(x)$ is $1/\alpha$. If the product $f'(x)g(x)$ of an arbitrary function $g(x)$ with $f'(x)$ is integrated over x, the value of the integral on the left-hand side of the x-axis will be $\alpha^{-1}g(-½\alpha)$, while that on the right-hand side will be $-\alpha^{-1}g(½\alpha)$. The total integral of $f'(x)g(x)$ will thus be $[g(-½\alpha) - g(½\alpha)]/\alpha$, which approaches $-g'(0)$ as $\alpha \to 0$. This is the expected sifting behavior of $\delta'(x)$. Therefore, in the limit when $\alpha \to 0$, the function $f'(x)$ approaches $\delta'(x)$.

c) Differentiating $f'(x)$ with respect to x, we find



$$f''(x) = \begin{cases} 0, & x \le -\alpha; \\ 4/\alpha^3, & -\alpha \le x \le -½\alpha; \\ -4/\alpha^3, & -½\alpha \le x \le ½\alpha; \\ 4/\alpha^3, & ½\alpha \le x \le \alpha; \\ 0, & x \ge \alpha. \end{cases}$$

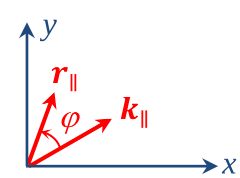
The area under each segment of the function $f''(x)$ is $2/\alpha^2$. If the product $f''(x)g(x)$ of an arbitrary function $g(x)$ with $f''(x)$ is integrated over x, the values of the integral over the four segments of $f''(x)$ will be, from left to right, $(2/\alpha^2)g(-¾\alpha)$, $-(2/\alpha^2)g(-¼\alpha)$, $-(2/\alpha^2)g(¼\alpha)$ and $(2/\alpha^2)g(¾\alpha)$. The total integral of $f''(x)g(x)$ will thus be

$$\frac{g(-¾\alpha)-g(-¼\alpha)-g(¼\alpha)+g(¾\alpha)}{\alpha^2/2} = \frac{\frac{g(¾\alpha)-g(¼\alpha)}{\alpha/2} - \frac{g(-¼\alpha)-g(-¾\alpha)}{\alpha/2}}{\alpha} \underset{\alpha\to 0}{\longrightarrow} \frac{g'(½\alpha)-g'(-½\alpha)}{\alpha} \underset{\alpha\to 0}{\longrightarrow} g''(0).$$

Since this is the expected sifting behavior of $\delta''(x)$, we conclude that, in the limit when $\alpha \to 0$, the function $f''(x)$ approaches $\delta''(x)$.

Problem 3) The Fourier transform of the charge distribution of the ring is straightforwardly evaluated, as follows:

$$\begin{aligned} \rho_{\text{free}}(\boldsymbol{k},\omega) &= \int_{-\infty}^{\infty} \rho_{\text{free}}(r_\parallel,\phi,z,t)\exp[-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,d\boldsymbol{r}dt \\ &= \int_{-\infty}^{\infty} \sigma_0 \left[\text{Circ}\left(\frac{r_\parallel}{R_2}\right) - \text{Circ}\left(\frac{r_\parallel}{R_1}\right)\right] \delta(z)\exp(-i\boldsymbol{k}_\parallel\cdot\boldsymbol{r}_\parallel)\exp(-ik_z z)\exp(i\omega t)\,d\boldsymbol{r}dt \\ &= 2\pi\sigma_0\delta(\omega)\int_{R_1}^{R_2}\int_0^{2\pi}\exp(-ik_\parallel r_\parallel\cos\varphi)\,r_\parallel dr_\parallel d\varphi \\ &= (2\pi)^2\sigma_0\,\delta(\omega)\int_{R_1}^{R_2} r_\parallel J_0(k_\parallel r_\parallel)\,dr_\parallel \end{aligned}$$



$$= (2\pi)^2\sigma_0\,\delta(\omega)k_\parallel^{-2}\int_{k_\parallel R_1}^{k_\parallel R_2} x\,J_0(x)\,dx$$

$$= (2\pi)^2\sigma_0\,\delta(\omega)k_\parallel^{-1}[R_2 J_1(k_\parallel R_2) - R_1 J_1(k_\parallel R_1)].$$

Problem 4) a) Transforming the equations to the Fourier domain, we find

$$i\varepsilon_o \boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{k},\omega) = \rho_{\text{free}}(\boldsymbol{k},\omega) - i\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega), \tag{1a}$$

$$i\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega) = \boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - i\omega\boldsymbol{P}(\boldsymbol{k},\omega) - i\varepsilon_o\omega\boldsymbol{E}(\boldsymbol{k},\omega), \tag{1b}$$

$$i\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega) = i\omega\boldsymbol{M}(\boldsymbol{k},\omega) + i\mu_o\omega\boldsymbol{H}(\boldsymbol{k},\omega), \tag{1c}$$

$$i\mu_o\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega) = -i\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega). \tag{1d}$$

b) Cross-multiplying $\boldsymbol{k}$ into Eqs.(1b) and (1c) yields

$$\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)] = -i\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - \omega\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega) - \varepsilon_o\omega\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega), \tag{2a}$$

$$\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega)] = \omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega) + \mu_o\omega\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega). \tag{2b}$$

The above equations may now be simplified if $\boldsymbol{k}\times\boldsymbol{E}$ from Eq.(1c) and $\boldsymbol{k}\times\boldsymbol{H}$ from Eq.(1b) are substituted into Eqs.(2a) and Eq.(2b), respectively, and also if the vector identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$ is used to simplify the left-hand sides of Eqs.(2a) and (2b), as follows:

$$[\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega)]\boldsymbol{k} - k^2\boldsymbol{H}(\boldsymbol{k},\omega) = -i\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - \omega\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega) - \varepsilon_o\omega^2[\boldsymbol{M}(\boldsymbol{k},\omega) + \mu_o\boldsymbol{H}(\boldsymbol{k},\omega)], \tag{3a}$$

$$[\boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{k},\omega)]\boldsymbol{k} - k^2\boldsymbol{E}(\boldsymbol{k},\omega) = \omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega) - i\mu_o\omega\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - \mu_o\omega^2[\boldsymbol{P}(\boldsymbol{k},\omega) + \varepsilon_o\boldsymbol{E}(\boldsymbol{k},\omega)]. \tag{3b}$$

Next, we substitute $\boldsymbol{k}\cdot\boldsymbol{H}$ from Eq.(1d) into Eq.(3a), and $\boldsymbol{k}\cdot\boldsymbol{E}$ from Eq.(1a) into Eq.(3b) to obtain

$$[(\omega/c)^2 - k^2]\boldsymbol{H}(\boldsymbol{k},\omega) = -i\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) - \omega\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega) - \varepsilon_o\omega^2\boldsymbol{M}(\boldsymbol{k},\omega) + \mu_o^{-1}[\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)]\boldsymbol{k}, \tag{4a}$$

$$[(\omega/c)^2 - k^2]\boldsymbol{E}(\boldsymbol{k},\omega) = i\varepsilon_o^{-1}\rho_{\text{free}}(\boldsymbol{k},\omega)\boldsymbol{k} - i\mu_o\omega\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) + \omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega) - \mu_o\omega^2\boldsymbol{P}(\boldsymbol{k},\omega) + \varepsilon_o^{-1}[\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k}. \tag{4b}$$

The final solutions for the electromagnetic fields $\boldsymbol{E}(\boldsymbol{k},\omega)$ and $\boldsymbol{H}(\boldsymbol{k},\omega)$ are thus given by

$$\boldsymbol{H}(\boldsymbol{k},\omega) = \frac{i\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) + \omega\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega) + \varepsilon_o\omega^2\boldsymbol{M}(\boldsymbol{k},\omega) - \mu_o^{-1}[\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)]\boldsymbol{k}}{k^2 - (\omega/c)^2}, \tag{5a}$$

$$\boldsymbol{E}(\boldsymbol{k},\omega) = \frac{-i\varepsilon_o^{-1}\rho_{\text{free}}(\boldsymbol{k},\omega)\boldsymbol{k} + i\mu_o\omega\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega) + \mu_o\omega^2\boldsymbol{P}(\boldsymbol{k},\omega) - \varepsilon_o^{-1}[\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k} - \omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)}{k^2 - (\omega/c)^2}. \tag{5b}$$

It is not difficult to verify that the above expressions for E and H fields are the same as those obtained using the bound electric charge and current densities — with or without the introduction of scalar and vector potentials.
