Problem 1) a) $J_{\text{bound}}^{(e)} = \mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t) = \mu_o^{-1}\boldsymbol{\nabla}\times[m_o\delta(x)\delta(y)\delta(z)\hat{\boldsymbol{z}}]$

$$= \mu_o^{-1}m_o[\delta(x)\delta'(y)\hat{\boldsymbol{x}} - \delta'(x)\delta(y)\hat{\boldsymbol{y}}]\delta(z).$$

b) $\boldsymbol{A}(\boldsymbol{r}) = (\mu_o/4\pi)\int_{-\infty}^{\infty}[\boldsymbol{J}(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|]\mathrm{d}\boldsymbol{r}'$

$$= \frac{m_o}{4\pi}\int_{-\infty}^{\infty}\frac{[\delta(x')\delta'(y')\hat{\boldsymbol{x}} - \delta'(x')\delta(y')\hat{\boldsymbol{y}}]\delta(z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \quad \leftarrow \boxed{\text{Sifting property of } \delta(\cdot)}$$

$$= \frac{m_o\hat{\boldsymbol{x}}}{4\pi}\int_{-\infty}^{\infty}\frac{\delta'(y')}{\sqrt{x^2+(y-y')^2+z^2}}\mathrm{d}y' - \frac{m_o\hat{\boldsymbol{y}}}{4\pi}\int_{-\infty}^{\infty}\frac{\delta'(x')}{\sqrt{(x-x')^2+y^2+z^2}}\mathrm{d}x' \quad \leftarrow \boxed{\text{Sifting property of } \delta'(\cdot)}$$

$$= -\frac{m_o\hat{\boldsymbol{x}}}{4\pi}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{m_o\hat{\boldsymbol{y}}}{4\pi}\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{m_o(x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}})}{4\pi(x^2+y^2+z^2)^{3/2}} = \frac{m_o\hat{\boldsymbol{z}}\times\boldsymbol{r}}{4\pi r^3} = \frac{m_o[(\cos\theta)\hat{\boldsymbol{r}} - (\sin\theta)\hat{\boldsymbol{\theta}}]\times\hat{\boldsymbol{r}}}{4\pi r^2} = \frac{m_o\sin\theta}{4\pi r^2}\hat{\boldsymbol{\phi}}.$$

c) $\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\left(\frac{m_o\sin\theta}{4\pi r^2}\hat{\boldsymbol{\phi}}\right) = \frac{m_o}{4\pi}\left[\frac{\partial(\sin^2\theta/r^2)}{r\sin\theta\,\partial\theta}\hat{\boldsymbol{r}} - \frac{\partial(\sin\theta/r)}{r\partial r}\hat{\boldsymbol{\theta}}\right]$

$$= \frac{m_o}{4\pi}\left[\frac{2\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right] = \frac{m_o}{4\pi r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}).$$

Problem 2) a) The symmetry of the problem dictates that the potential be a function of the radial distance $\rho = \sqrt{x^2+y^2}$ from the wire. At an observation point located in the xy-plane at a distance ρ from the origin, we will have

$$\psi(\boldsymbol{r}) = (4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}[\rho(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|]\mathrm{d}\boldsymbol{r}'$$

$$= (4\pi\varepsilon_o)^{-1}\lim_{z_0\to\infty}\int_{-z_0}^{z_0}\frac{\lambda_o\delta(x')\delta(y')}{\sqrt{(x-x')^2+(y-y')^2+z'^2}}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$$

$$= \frac{\lambda_o}{2\pi\varepsilon_o}\lim_{z_0\to\infty}\int_0^{z_0}\frac{\mathrm{d}z'}{\sqrt{x^2+y^2+z'^2}} = \frac{\lambda_o}{2\pi\varepsilon_o}\lim_{z_0\to\infty}\ln\left(z'+\sqrt{\rho^2+z'^2}\right)\Big|_{z'=0}^{z_0} \quad \leftarrow \rho = \sqrt{x^2+y^2}$$

$$= \frac{\lambda_o}{2\pi\varepsilon_o}\left[\lim_{z_0\to\infty}\ln\left(z_0+\sqrt{\rho^2+z_0^2}\right) - \ln\rho\right].$$

The first term on the right-hand-side of the above expression is infinitely large, but it does not vary with ρ and may, therefore, be ignored. The scalar potential is thus given by

$$\psi(\boldsymbol{r}) = -(\lambda_o/2\pi\varepsilon_o)\ln\rho.$$

b) Since $\boldsymbol{E}=-\boldsymbol{\nabla}\psi$, we investigate the gradient of the neglected function $\ln(z_0+\sqrt{\rho^2+z_0^2})$ in the limit when $z_0\to\infty$, to see if it has any dependence on the radial distance ρ. We find

$$\frac{\partial}{\partial\rho}\ln\left(z_0+\sqrt{\rho^2+z_0^2}\right)=\frac{\rho/\sqrt{\rho^2+z_0^2}}{z_0+\sqrt{\rho^2+z_0^2}}=\frac{\rho/z_0^2}{1+(\rho/z_0)^2+\sqrt{1+(\rho/z_0)^2}}.$$

Thus, for any finite value of ρ, in the limit when $z_0\to\infty$, the denominator of the above expression approaches 2, while the numerator approaches zero. It is thus clear that, for sufficiently large z_0, the contribution to the E-field of $\ln(z_0+\sqrt{\rho^2+z_0^2})$ at any finite radial distance ρ is negligibly small.

c) The calculation of $\boldsymbol{A}(\boldsymbol{r})$ follows essentially the same steps as the above calculation of $\psi(\boldsymbol{r})$. We will have

$$\boldsymbol{A}(\boldsymbol{r})=(\mu_o/4\pi)\int_{-\infty}^{\infty}[\boldsymbol{J}(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|]\mathrm{d}\boldsymbol{r}'=(\mu_o/4\pi)\lim_{z_0\to\infty}\int_{-z_0}^{z_0}\frac{I_o\delta(x')\delta(y')\hat{\boldsymbol{z}}}{\sqrt{(x-x')^2+(y-y')^2+z'^2}}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$$

$$=\frac{\mu_o I_o\hat{\boldsymbol{z}}}{2\pi}\lim_{z_0\to\infty}\int_0^{z_0}\frac{\mathrm{d}z'}{\sqrt{x^2+y^2+z'^2}}=\frac{\mu_o I_o\hat{\boldsymbol{z}}}{2\pi}\lim_{z_0\to\infty}\ln\left(z'+\sqrt{\rho^2+z'^2}\right)\Big|_{z'=0}^{z_0}$$

$$=\frac{\mu_o I_o\hat{\boldsymbol{z}}}{2\pi}\left[\lim_{z_0\to\infty}\ln\left(z_0+\sqrt{\rho^2+z_0^2}\right)-\ln\rho\right].$$

Ignoring the first term on the right-hand-side of the above equation, we find the vector potential of the wire to be $\boldsymbol{A}(\boldsymbol{r})=-(\mu_o I_o\hat{\boldsymbol{z}}/2\pi)\ln\rho$.

Problem 3)

a) $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=0,$ ← $\rho_{\text{free}}(\boldsymbol{r},t)$ is set to zero.

$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t,$ ← $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)$ is set to zero.

$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t \quad\to\quad \boldsymbol{\nabla}\times(\varepsilon_o\boldsymbol{E}+\boldsymbol{P})=-\varepsilon_o(\partial\boldsymbol{M}/\partial t-\varepsilon_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{P})-\mu_o\varepsilon_o\partial\boldsymbol{H}/\partial t$

$\to\quad \boldsymbol{\nabla}\times\boldsymbol{D}(\boldsymbol{r},t)=-\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}-\mu_o\varepsilon_o\partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t \quad\leftarrow \boldsymbol{J}_{\text{bound}}^{(m)}=\partial\boldsymbol{M}/\partial t-\varepsilon_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{P},$

$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0 \quad\to\quad \mu_o\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r},t)=\rho_{\text{bound}}^{(m)} \quad\leftarrow \rho_{\text{bound}}^{(m)}=-\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r},t).$

b) Since Maxwell's 1st equation ensures that $\boldsymbol{\nabla}\cdot\boldsymbol{D}=0$, we define the magnetic vector potential $\boldsymbol{A}^{(m)}(\boldsymbol{r},t)$ such that $\boldsymbol{D}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\times\boldsymbol{A}^{(m)}(\boldsymbol{r},t)$. Substitution into Maxwell's 2nd equation yields

$\boldsymbol{\nabla}\times[\boldsymbol{H}(\boldsymbol{r},t)+\partial\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t]=0 \quad\to\quad \boldsymbol{H}(\boldsymbol{r},t)+\partial\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t=-\boldsymbol{\nabla}\psi^{(m)}(\boldsymbol{r},t).$ ← Because $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\psi^{(m)}=0$.

c) Using the above magnetic potentials, Maxwell's 3rd equation may be written as follows:

$$\boldsymbol{\nabla}\times[-\boldsymbol{\nabla}\times\boldsymbol{A}^{(m)}(\boldsymbol{r},t)]=-\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}-\mu_o\varepsilon_o(\partial/\partial t)[-\boldsymbol{\nabla}\psi^{(m)}(\boldsymbol{r},t)-\partial\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t]$$

$$\to\quad \boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot\boldsymbol{A}^{(m)}(\boldsymbol{r},t)]-\boldsymbol{\nabla}^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)=\varepsilon_o\boldsymbol{J}_{\text{bound}}^{(m)}-(1/c^2)\boldsymbol{\nabla}[\partial\psi^{(m)}(\boldsymbol{r},t)/\partial t]-(1/c^2)\partial^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t^2$$

$$\rightarrow \quad \boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot\boldsymbol{A}^{(m)}(\boldsymbol{r},t)+(1/c^2)\partial\psi^{(m)}(\boldsymbol{r},t)/\partial t]-\boldsymbol{\nabla}^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)+(1/c^2)\partial^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t^2=\varepsilon_o\boldsymbol{J}^{(m)}_{\text{bound}}.$$

Clearly, eliminating $\psi^{(m)}$ from the above equation requires the same gauge for the magnetic potentials as the Lorenz gauge that was defined for the electric potentials, that is,

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}^{(m)}(\boldsymbol{r},t)+(1/c^2)\partial\psi^{(m)}(\boldsymbol{r},t)/\partial t=0.$$ ← Equivalent of Lorenz gauge

d) Maxwell's 3rd equation in conjunction with the Lorenz-equivalent gauge now becomes

$$\boldsymbol{\nabla}^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)-(1/c^2)\partial^2\boldsymbol{A}^{(m)}(\boldsymbol{r},t)/\partial t^2=-\varepsilon_o\boldsymbol{J}^{(m)}_{\text{bound}}.$$

This wave equation for the magnetic vector potential, having $\varepsilon_o\boldsymbol{J}^{(m)}_{\text{bound}}$ for the source term, is the counterpart of the equation for the standard vector potential, where the source term is $\mu_o\boldsymbol{J}^{(e)}_{\text{total}}$.

The wave equation for $\psi^{(m)}(\boldsymbol{r},t)$ could similarly be derived by substituting into Maxwell's 4th equation the expressions that relate $\boldsymbol{D}$ and $\boldsymbol{H}$ to $\boldsymbol{A}^{(m)}$ and $\psi^{(m)}$. The final result, after taking account of the Lorenz-equivalent gauge, is

$$\mu_o\boldsymbol{\nabla}\cdot[-\boldsymbol{\nabla}\psi^{(m)}-\partial\boldsymbol{A}^{(m)}/\partial t]=\rho^{(m)}_{\text{bound}} \quad\rightarrow\quad \boldsymbol{\nabla}^2\psi^{(m)}(\boldsymbol{r},t)-(1/c^2)\partial^2\psi^{(m)}(\boldsymbol{r},t)/\partial t^2=-\mu_o^{-1}\rho^{(m)}_{\text{bound}}(\boldsymbol{r},t).$$

Once again we have an equation similar to the standard wave equation for the (electric) scalar potential, except that the source term here is $\mu_o^{-1}\rho^{(m)}_{\text{bound}}$, rather than $\varepsilon_o^{-1}\rho^{(e)}_{\text{total}}$.

e) By analogy with the standard wave equation, we write the solutions to the preceding wave equations for magnetic potentials as follows:

$$\psi^{(m)}(\boldsymbol{r},t)=(1/4\pi\mu_o)\int_{-\infty}^{\infty}\frac{\rho^{(m)}_{\text{bound}}(\boldsymbol{r}',t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'.$$

$$\boldsymbol{A}^{(m)}(\boldsymbol{r},t)=(\varepsilon_o/4\pi)\int_{-\infty}^{\infty}\frac{\boldsymbol{J}^{(m)}_{\text{bound}}(\boldsymbol{r}',t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}';$$