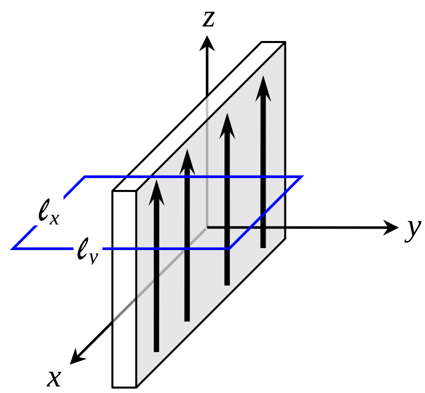


Problem 1) a) Because of symmetry, the H-field cannot depend on x or z. Take a rectangular loop $\ell_x \times \ell_y$ parallel to the xy-plane and write the integral form of Ampere's law, $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}}$, for this loop. The contributions of ℓ_y to the loop integral cancel out, leaving only the contributions of ℓ_x on opposite sides of the current sheet. Therefore, $2H_x\ell_x = J_{so}\ell_x$, where $J_{so}\ell_x$ is the current crossing the loop. The magnitude of the H-field is thus independent of y, although its direction depends on whether y is positive or negative. Taking the right-hand rule into account, the final result is

$$\boldsymbol{H}(\boldsymbol{r},t) = -\tfrac{1}{2}\operatorname{sign}(y)\, J_{so}\,\hat{\boldsymbol{x}}.$$

Note: Using symmetry and Maxwell's 4th equation, $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, it is easy to see why H_y must be zero everywhere: Take a cylinder whose axis is parallel to y and which the xz-plane cuts in the middle, then use the fact that the net flux of $\boldsymbol{H}$ into or out of the cylinder must be zero. Similarly, Maxwell's 2nd equation can be used to show that H_z is independent of y; the argument parallels that used above to evaluate H_x, except that the rectangular loop is now chosen in the yz-plane. Since H_z is already known to be independent of x and z, we conclude that it must be constant through the entire space. Showing that H_z is identically zero, however, requires the full solution of Maxwell's equations, which is done in part (b).

b) Fourier transforming the current density $\boldsymbol{J}(\boldsymbol{r},t) = J_{so}\delta(y)\,\hat{\boldsymbol{z}}$ yields

$$\boldsymbol{J}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty} J_{so}\delta(y)\,\hat{\boldsymbol{z}}\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\mathrm{d}\boldsymbol{r}\,\mathrm{d}t = (2\pi)^3 J_{so}\,\delta(k_x)\,\delta(k_z)\,\delta(\omega)\hat{\boldsymbol{z}}.$$

The H-field is thus given by

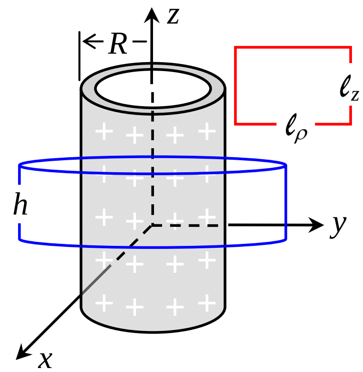
$$\boldsymbol{H}(\boldsymbol{r},t) = \mu_o^{-1}\boldsymbol{B}(\boldsymbol{r},t) = \mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t) = (2\pi)^{-4}\int_{-\infty}^{\infty}\frac{\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}(\boldsymbol{k},\omega)}{k^2-(\omega/c)^2}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$

$$= \frac{\mathrm{i}J_{so}}{2\pi}\int_{-\infty}^{\infty}\frac{(\boldsymbol{k}\times\hat{\boldsymbol{z}})\,\delta(k_x)\delta(k_z)\delta(\omega)}{k^2-(\omega/c)^2}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\mathrm{d}\boldsymbol{k}\mathrm{d}\omega$$

$$= \frac{\mathrm{i}J_{so}}{2\pi}\int_{-\infty}^{\infty}\frac{k_y\hat{\boldsymbol{y}}\times\hat{\boldsymbol{z}}}{k_y^2}\exp(\mathrm{i}k_y y)\,\mathrm{d}k_y = \mathrm{i}J_{so}\hat{\boldsymbol{x}}\,(2\pi)^{-1}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}k_y y)}{k_y}\mathrm{d}k_y = -\tfrac{1}{2}\operatorname{sign}(y)J_{so}\hat{\boldsymbol{x}}.$$

Problem 2) a) Because of symmetry, the E-field is independent of ϕ and z. Take a cylinder of radius ρ and height h, and write the integral form of Maxwell's first equation, $\boldsymbol{\nabla}\cdot\varepsilon_o\boldsymbol{E} = \rho_{\text{free}}$, for this cylinder. The contributions to the integral of the top and bottom surfaces of the cylinder cancel out, leaving only the contribution of the cylindrical side-wall, which is $2\pi\rho h\,\varepsilon_o E_\rho(\rho)$. Therefore, $2\pi\rho h\,\varepsilon_o E_\rho(\rho) = 2\pi Rh\sigma_{so}$, where the right-hand-side of the equation gives the total electrical charge inside the cylinder of radius ρ, provided, of course, that $\rho > R$. Consequently,

$E_\rho(\rho) = R\sigma_{so}/(\rho\,\varepsilon_o)$ when $\rho > R$, and $E_\rho(\rho) = 0$ when $\rho < R$. From Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E} = 0$, we conclude that $E_\phi = 0$, otherwise a circular loop of radius ρ, parallel to the xy-plane and centered on the z-axis, will have a nonzero line integral. As for E_z, consider the rectangular loop $\ell_\rho \times \ell_z$ shown in the figure. The contributions of ℓ_ρ to the line-integral of the E-field around the loop cancel out because E_ρ is independent of z. For the contributions of the vertical legs, ℓ_z, to also cancel out, it is necessary for E_z to be independent of ρ. We thus see that E_z must be constant through the entire space. In fact, because of the system's up-down symmetry, it is not difficult to see that E_z must be identically zero everywhere: There is as much reason for the E-field to point up as there is for it to point down. Therefore $E_z = 0$ and we have



$$\boldsymbol{E}(\boldsymbol{r},t) = \begin{cases} (R\sigma_{so}/\varepsilon_o\rho)\hat{\boldsymbol{\rho}}; & \rho > R, \\ 0; & \rho < R. \end{cases}$$

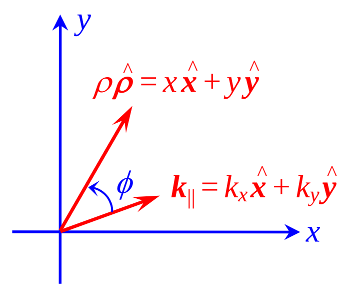
b) The Fourier transform of the electric charge-density, $\rho(\boldsymbol{r},t) = \sigma_{so}\delta(\rho - R)$, is given by

$$\rho(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty} \sigma_{so}\delta(\rho - R)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\mathrm{d}\boldsymbol{r}\,\mathrm{d}t$$

$$= (2\pi)^2\sigma_{so}\,\delta(k_z)\,\delta(\omega)\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi}\delta(\rho - R)\exp(-\mathrm{i}\,k_\parallel\rho\cos\phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$

$$= (2\pi)^2 R\sigma_{so}\,\delta(k_z)\,\delta(\omega)\int_{\phi=0}^{2\pi}\exp(-\mathrm{i}\,k_\parallel R\cos\phi)\,\mathrm{d}\phi$$

$$= (2\pi)^3 R\sigma_{so}\,\delta(k_z)\,\delta(\omega)\,J_0(k_\parallel R).$$



The E-field is thus obtained as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t) = -(2\pi)^{-4}\int_{-\infty}^{\infty}\frac{\mathrm{i}\boldsymbol{k}\rho(\boldsymbol{k},\omega)}{\varepsilon_o[k^2 - (\omega/c)^2]}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$= -\frac{\mathrm{i}R\sigma_{so}}{2\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{\boldsymbol{k}\,\delta(k_z)\,\delta(\omega)\,J_0(k_\parallel R)}{k^2 - (\omega/c)^2}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$= -\frac{\mathrm{i}R\sigma_{so}}{2\pi\varepsilon_o}\int_{k_\parallel=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{k_\parallel\cos\phi\hat{\boldsymbol{\rho}}\,J_0(k_\parallel R)}{k_\parallel^2}\exp(\mathrm{i}k_\parallel\rho\cos\phi)\,k_\parallel\mathrm{d}k_\parallel\mathrm{d}\phi$$

$$= -\frac{\mathrm{i}R\sigma_{so}\hat{\boldsymbol{\rho}}}{2\pi\varepsilon_o}\int_{k_\parallel=0}^{\infty}J_0(k_\parallel R)\int_{\phi=0}^{2\pi}\cos\phi\,\exp(\mathrm{i}k_\parallel\rho\cos\phi)\mathrm{d}\phi\,\mathrm{d}k_\parallel$$

$$= \frac{R\sigma_{so}\hat{\boldsymbol{\rho}}}{\varepsilon_o}\int_0^{\infty}J_0(k_\parallel R)\,J_1(k_\parallel\rho)\mathrm{d}k_\parallel = \begin{cases} (R\sigma_{so}/\varepsilon_o\rho)\hat{\boldsymbol{\rho}}; & \rho > R, \\ 0; & \rho < R. \end{cases}$$
