Problem 1) a) $\vec E_0 = \vec E_0' + i\vec E_0''$. When $\vec E_0'$ and $\vec E_0''$ are <u>parallel</u> to each other, i.e., when they point in the same direction, the beam is <u>linearly polarized</u>. Let's define the unit vector $\hat u$ as the common direction of $\vec E_0'$ and $\vec E_0''$. We'll have:

$$\vec E(\vec r,t) = \vec E_0' \cos(\vec k\cdot\vec r - \omega t) + \vec E_0'' \sin(\vec k\cdot\vec r-\omega t) = \sqrt{E_0'^2+E_0''^2}\,[\cos\phi_0 \cos(\vec k\cdot\vec r-\omega t)$$
$$+ \sin\phi_0 \sin(\vec k\cdot\vec r-\omega t)]\,\hat u = \sqrt{E_0'^2+E_0''^2}\,\cos[\vec k\cdot\vec r-\omega t+\phi_0]\,\hat u$$

The last expression represents a linearly-polarized plane-wave, with E-field along the $\hat u$ direction. The phase-angle ϕ_0 is related to E_0' and E_0'' as follows: $\tan\phi_0 = E_0''/E_0'$.

b) $\vec E_0 = \vec E_0' + i\vec E_0''$. The plane-wave is circularly-polarized when $\vec E_0' \perp \vec E_0''$ <u>and</u> $|\vec E_0'| = |\vec E_0''|$. At a fixed point in space, say $\vec r = \vec r_0$, when $t = \vec k\cdot\vec r_0/\omega$ we have $\vec E(\vec r_0,t) = \vec E_0'$. A quarter of a period later (Note: Period $T = 2\pi/\omega$), when $t = \frac14 T + (\vec k\cdot\vec r_0/\omega)$, we'll have $\vec E(\vec r_0,t) = \vec E_0''$. By definition, the E-field of a circularly-polarized wave rotates uniformly at frequency ω, covering a quarter of the circle during each quarter-period, $T/4$. Therefore, $\vec E_0'$ and $\vec E_0''$ must have equal length and be perpendicular to each other.

c) Maxwell's equations in free-space:

1) $\vec\nabla\cdot\vec E = 0 \Rightarrow \vec k\cdot\vec E_0 = 0$

2) $\vec\nabla\times\vec H = \epsilon_0\,\partial\vec E/\partial t \Rightarrow \vec k\times\vec H_0 = -\epsilon_0\omega\vec E_0$

3) $\vec\nabla\times\vec E = -\mu_0\,\partial\vec H/\partial t \Rightarrow \vec k\times\vec E_0 = \mu_0\omega\vec H_0$

4) $\vec\nabla\cdot\vec H = 0 \Rightarrow \vec k\cdot\vec H_0 = 0$

From 2) and 3): $\vec k\times(\vec k\times\vec E_0) = -\mu_0\epsilon_0\omega^2\vec E_0 \Rightarrow \underbrace{\cancelto{0}{(\vec k\cdot\vec E_0)}}_{\text{Maxwell's Eq. (1)}}\vec k - (\vec k\cdot\vec k)\vec E_0 = -(\omega/c)^2\vec E_0 \Rightarrow k^2 = (\omega/c)^2$

Since $\vec k$ for a homogeneous plane-wave is real valued, we have:

$$k^2 = \vec k\cdot\vec k = (\vec k' + i\vec k'')\cdot(\vec k'+i\vec k'') = k'^2 - k''^2 + 2i\,\vec k'\cdot\vec k'' = k'^2 \leftarrow \text{real}$$

Therefore, $k^2 = (\omega/c)^2 \Rightarrow k = \omega/c$.

d) From Maxwell's equation 1, using the fact that $\vec{k} = \vec{k}' + i\vec{k}'' = \vec{k}'$, i.e., $\vec{k}$ is real-valued, we write:

$$\vec{k}\cdot\vec{E}_0 = 0 \Rightarrow \vec{k}\cdot(\vec{E}_0' + i\vec{E}_0'') = 0 \Rightarrow \vec{k}\cdot\vec{E}_0' + i\vec{k}\cdot\vec{E}_0'' = 0 \Rightarrow \begin{cases} \vec{k}\cdot\vec{E}_0' = 0 \\ \vec{k}\cdot\vec{E}_0'' = 0 \end{cases}.$$

Therefore, $\vec{k} \perp \vec{E}_0'$ and $\vec{k} \perp \vec{E}_0''$.

e) From Maxwell's 4th equation $\vec{k}\cdot\vec{H}_0 = 0 \Rightarrow \vec{k}\cdot(\vec{H}_0' + i\vec{H}_0'') = 0 \Rightarrow \begin{cases} \vec{k}\cdot\vec{H}_0' = 0 \\ \vec{k}\cdot\vec{H}_0'' = 0 \end{cases}$.

Therefore, $\vec{k} \perp \vec{H}_0'$ and $\vec{k} \perp \vec{H}_0''$.

f) From Maxwell's 3rd equation $\vec{k}\times\vec{E}_0 = \mu_0\omega\vec{H}_0 \Rightarrow \vec{k}\times\vec{E}_0' + i\vec{k}\times\vec{E}_0'' = \mu_0\omega(\vec{H}_0' + i\vec{H}_0'')$

$$\Rightarrow \begin{cases} \vec{H}_0' = (\vec{k}\times\vec{E}_0')/\mu_0\omega \\ \vec{H}_0'' = (\vec{k}\times\vec{E}_0'')/\mu_0\omega \end{cases} \xrightarrow[\text{also } \vec{k}\perp\vec{E}_0'']{\text{Since } \vec{k}\perp\vec{E}_0' \text{ and}} \begin{cases} \vec{H}_0' \perp \vec{E}_0' \text{ and } H_0' = \frac{\omega/c}{\mu_0\omega}E_0' = E_0'/Z_0. \\ \vec{H}_0'' \perp \vec{E}_0'' \text{ and } H_0'' = \frac{\omega/c}{\mu_0\omega}E_0'' = E_0''/Z_0. \end{cases}$$

g) $$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t) = [\vec{E}_0'\cos(\vec{k}\cdot\vec{r}-\omega t) - \vec{E}_0''\sin(\vec{k}\cdot\vec{r}-\omega t)]\times[\vec{H}_0'\cos(\vec{k}\cdot\vec{r}-\omega t) - \vec{H}_0''\sin(\vec{k}\cdot\vec{r}-\omega t)]$$

$$= (\vec{E}_0'\times\vec{H}_0')\cos^2(\vec{k}\cdot\vec{r}-\omega t) + (\vec{E}_0''\times\vec{H}_0'')\sin^2(\vec{k}\cdot\vec{r}-\omega t) - \tfrac{1}{2}(\vec{E}_0'\times\vec{H}_0'' + \vec{E}_0''\times\vec{H}_0')\sin[2(\vec{k}\cdot\vec{r}-\omega t)]$$

$$= \frac{E_0'^2}{Z_0}\hat{k}\cos^2(\vec{k}\cdot\vec{r}-\omega t) + \frac{E_0''^2}{Z_0}\hat{k}\sin^2(\vec{k}\cdot\vec{r}-\omega t) - \frac{1}{2Z_0}(\vec{E}_0'\cdot\vec{E}_0'' + \vec{E}_0''\cdot\vec{E}_0')\hat{k}\sin[2(\vec{k}\cdot\vec{r}-\omega t)]$$

$$\Rightarrow \vec{S}(\vec{r},t) = \frac{E_0'^2 + E_0''^2}{2Z_0}\hat{k} + \frac{E_0'^2 - E_0''^2}{2Z_0}\hat{k}\cos[2(\vec{k}\cdot\vec{r}-\omega t)] - \frac{\vec{E}_0'\cdot\vec{E}_0''}{Z_0}\hat{k}\sin[2(\vec{k}\cdot\vec{r}-\omega t)] \Rightarrow$$

$$\vec{S}(\vec{r},t) = \frac{\hat{k}}{2Z_0}\left\{(E_0'^2 + E_0''^2) + (E_0'^2 - E_0''^2)\cos\left[2\omega\left(t - \frac{\hat{k}\cdot\vec{r}}{c}\right)\right] + 2\vec{E}_0'\cdot\vec{E}_0''\sin\left[2\omega\left(t - \frac{\hat{k}\cdot\vec{r}}{c}\right)\right]\right\}$$

The energy propagates along $\hat{k}$.

Problem 2) Maxwell's equations in free space:

1) $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0$
2) $\vec{\nabla}\times\vec{H} = \epsilon_0\partial\vec{E}/\partial t \Rightarrow \vec{k}\times\vec{H}_0 = -\epsilon_0\omega\vec{E}_0$
3) $\vec{\nabla}\times\vec{E} = -\mu_0\partial\vec{H}/\partial t \Rightarrow \vec{k}\times\vec{E}_0 = +\mu_0\omega\vec{H}_0$
4) $\vec{\nabla}\cdot\vec{H} = 0 \Rightarrow \vec{k}\cdot\vec{H}_0 = 0$

(2), (3) $\Rightarrow \vec{k}\times(\vec{k}\times\vec{E}_0) = -\mu_0\epsilon_0\omega^2\vec{E}_0 \Rightarrow$

$$\underbrace{\cancel{(\vec{k}\cdot\vec{E}_0)}}_{=0 \text{ from Eq.(1)}}\vec{k} - k^2\vec{E}_0 = -(\omega/c)^2\vec{E}_0 \Rightarrow k^2 = (\omega/c)^2$$

a) $\vec{k} = k_x \hat{x} + i k_z \hat{z} \Rightarrow k^2 = \vec{k}\cdot\vec{k} = (k_x \hat{x} + i k_z \hat{z})\cdot(k_x \hat{x} + i k_z \hat{z}) = k_x^2 - k_z^2 = (\omega/c)^2.$

b) From Maxwell's 1st equation: $\vec{k}\cdot\vec{E}_0 = 0 \Rightarrow (k_x \hat{x} + i k_z \hat{z})\cdot(E_{x0}\hat{x} + i E_{z0}\hat{z}) = 0 \Rightarrow$

$k_x E_{x0} - k_z E_{z0} = 0 \Rightarrow k_x E_{x0} = k_z E_{z0}.$

c) From Maxwell's 3rd Equation: $\vec{k}\times\vec{E}_0 = \mu_0 \omega \vec{H}_0 \Rightarrow (k_x \hat{x} + i k_z \hat{z})\times(E_{x0}\hat{x} + i E_{z0}\hat{z})$

$= i\mu_0 \omega H_{y0}\hat{y} \Rightarrow i k_x E_{z0}(\hat{x}\times\hat{z}) + i k_z E_{x0}(\hat{z}\times\hat{x}) = i(k_z E_{x0} - k_x E_{z0})\hat{y} = i\mu_0\omega H_{y0}\hat{y}$

$$\Rightarrow H_{y0} = \frac{k_z E_{x0} - k_x E_{z0}}{\mu_0 \omega} = \frac{k_z - (k_x^2/k_z)}{\mu_0\omega} E_{x0} = \frac{k_z^2 - k_x^2}{\mu_0 \omega k_z} E_{x0} = -\frac{(\omega/c)^2}{\mu_0\omega\sqrt{k_x^2 - \omega^2/c^2}} E_{x0}$$

$$\Rightarrow H_{y0} = -\frac{\omega/c}{\mu_0\omega\sqrt{(ck_x/\omega)^2 - 1}} E_{x0} \Rightarrow H_{y0} = -\frac{E_{x0}}{Z_0\sqrt{(\frac{ck_x}{\omega})^2 - 1}}$$

In the above derivation we chose the positive sign for $k_z = \sqrt{k_x^2 - (\omega/c)^2}$, the reason being that, the field-amplitudes are proportional to $\exp(i\vec{k}\cdot\vec{r}) = \exp[i(k_x\hat{x} + i k_z\hat{z})\cdot\vec{r}] = \exp(-k_z z)\exp(ik_x x)$. When $z \to +\infty$ the fields can't become indefinitely large; therefore, $k_z \geq 0$.

d) $\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}\{\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\} = \frac{1}{2}\mathrm{Re}\{(E_{x0}\hat{x} + iE_{z0}\hat{z})e^{i(k_x\hat{x}+ik_z\hat{z})\cdot\vec{r}} \times$

$(-iH_{y0}\hat{y})e^{-i(k_x\hat{x} - ik_z\hat{z})\cdot\vec{r}}\} = \frac{1}{2}e^{-2k_z z}\mathrm{Re}\{(E_{x0}\hat{x} + iE_{z0}\hat{z})\times(-iH_{y0}\hat{y})\}$

$= \frac{1}{2}e^{-2k_z z}\mathrm{Re}\{-iE_{x0}H_{y0}\hat{z} - E_{z0}H_{y0}\hat{x}\} = \frac{-1}{2}\exp(-2k_z z)E_{z0}H_{y0}\hat{x}$

$$\Rightarrow \langle\vec{S}(\vec{r},t)\rangle = \frac{1}{2}\exp(-2k_z z)\frac{E_{x0}E_{z0}\hat{x}}{Z_0\sqrt{(ck_x/\omega)^2 - 1}} = \frac{k_x E_{x0}^2\exp(-2k_z z)}{2Z_0 k_z\sqrt{(ck_x/\omega)^2-1}}\hat{x} \Rightarrow$$

$$\langle\vec{S}(\vec{r},t)\rangle = \frac{(ck_x/\omega)\exp[-2(\omega/c)\sqrt{(ck_x/\omega)^2 - 1}\, z]}{2Z_0[(ck_x/\omega)^2 - 1]}E_{x0}^2\,\hat{x}.$$

The energy flows along the positive x-axis when $k_x > 0$.

Problem 3) Maxwell's equations in an isotropic, homogeneous, linear medium:

1) $\vec{\nabla}\cdot\vec{D}=0 \Rightarrow \vec{k}\cdot\vec{E}_0=0$

2) $\vec{\nabla}\times\vec{H}=\partial\vec{D}/\partial t \Rightarrow \vec{k}\times\vec{H}_0=-\omega\epsilon_0\epsilon\vec{E}_0$

3) $\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t \Rightarrow \vec{k}\times\vec{E}_0=\omega\mu_0\mu\vec{H}_0$

4) $\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{k}\cdot\vec{H}_0=0$

$$\Rightarrow \vec{k}\times(\vec{k}\times\vec{E}_0)=-\omega^2\mu_0\epsilon_0\mu\epsilon\vec{E}_0 \Rightarrow$$

$$\underbrace{(\cancel{\vec{k}\cdot\vec{E}_0})}_{=0 \;\leftarrow \text{ from Eq.(1)}}\vec{E}_0-k^2\vec{E}_0=-(\omega/c)^2\mu\epsilon\vec{E}_0$$

$$\Rightarrow k^2=\vec{k}\cdot\vec{k}=(\omega/c)^2\mu(\omega)\epsilon(\omega).$$

a) Since $\vec{k}$ is real-valued, and $\mu(\omega)$ and $\epsilon(\omega)$ are both real and positive, we have $k^2=(\vec{k}'+i\vec{k}'')\cdot(\vec{k}'+i\vec{k}'')=k'^2-k''^2+2i\vec{k}'\cdot\vec{k}''=k'^2=(\omega/c)^2\mu(\omega)\epsilon(\omega)$

$$\Rightarrow k=|\vec{k}|=(\omega/c)\sqrt{\mu(\omega)\epsilon(\omega)}.$$

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b) From Maxwell's first equation $\vec{k}\cdot\vec{E}_0=0$. Since $\vec{k}$ is real-valued we have $\vec{k}\cdot(\vec{E}_0'+i\vec{E}_0'')=0 \Rightarrow \vec{k}\cdot\vec{E}_0'=0$ and $\vec{k}\cdot\vec{E}_0''=0$. Therefore, $\vec{k}\perp\vec{E}_0'$ and $\vec{k}\perp\vec{E}_0''$

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c) From Maxwell's fourth equation $\vec{k}\cdot\vec{H}_0=0$. Since $\vec{k}$ is real-valued we have: $\vec{k}\cdot(\vec{H}_0'+i\vec{H}_0'')=0 \Rightarrow \vec{k}\cdot\vec{H}_0'=0$ and $\vec{k}\cdot\vec{H}_0''=0$. Therefore, $\vec{k}\perp\vec{H}_0'$ and $\vec{k}\perp\vec{H}_0''$.

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d) From Maxwell's third equation $\vec{k}\times\vec{E}_0=\omega\mu_0\mu\vec{H}_0 \Rightarrow$ Since $\vec{k}$ is real-valued and also because $\mu(\omega)$ is real-valued

$$\vec{k}\times\vec{E}_0'+i\vec{k}\times\vec{E}_0''=\omega\mu_0\mu(\vec{H}_0'+i\vec{H}_0'') \Rightarrow$$

$$\left\{\begin{array}{l} \vec{H}_0'=\dfrac{\vec{k}\times\vec{E}_0'}{\omega\mu_0\mu} \\[2ex] \vec{H}_0''=\dfrac{\vec{k}\times\vec{E}_0''}{\omega\mu_0\mu}\end{array}\right. \xrightarrow[\text{and also } \vec{k}\perp\vec{E}_0'']{\text{Since } \vec{k}\perp\vec{E}_0'} \left\{\begin{array}{l} \vec{H}_0'\perp\vec{E}_0' \text{ and } H_0'=\dfrac{(\omega/c)\sqrt{\mu\epsilon}}{\omega\mu_0\mu}E_0'=\dfrac{E_0'}{Z_0\sqrt{\mu/\epsilon}} \\[2ex] \vec{H}_0''\perp\vec{E}_0'' \text{ and } H_0''=\dfrac{E_0''}{Z_0\sqrt{\mu/\epsilon}}\end{array}\right.$$

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e) $\vec{S}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)=[\vec{E}_0'\cos(\vec{k}\cdot\vec{r}-\omega t)-\vec{E}_0''\sin(\vec{k}\cdot\vec{r}-\omega t)]\times[\vec{H}_0'\cos(\vec{k}\cdot\vec{r}-\omega t)-\vec{H}_0''\sin(\vec{k}\cdot\vec{r}-\omega t)]$

$=(\vec{E}_0'\times\vec{H}_0')\cos^2(\vec{k}\cdot\vec{r}-\omega t)+(\vec{E}_0''\times\vec{H}_0'')\sin^2(\vec{k}\cdot\vec{r}-\omega t)-\frac{1}{2}(\vec{E}_0'\times\vec{H}_0''+\vec{E}_0''\times\vec{H}_0')\sin[2(\vec{k}\cdot\vec{r}-\omega t)]$

$$= \frac{E_0'^2}{Z_0\sqrt{\mu\epsilon}}\hat{k}\cos^2(\vec{k}\cdot\vec{r}-\omega t) + \frac{E_0''^2}{Z_0\sqrt{\mu\epsilon}}\hat{k}\sin^2(\vec{k}\cdot\vec{r}-\omega t) - \frac{\vec{E}_0'\cdot\vec{E}_0''+\vec{E}_0''\cdot\vec{E}_0'}{2Z_0\sqrt{\mu\epsilon}}\hat{k}\sin[2(\vec{k}\cdot\vec{r}-\omega t)].$$

Defining the real and positive refractive index $n(\omega)=\sqrt{\mu(\omega)\epsilon(\omega)}$, we can rewrite the above expression as follows;

$$\vec{S}(\vec{r},t) = \frac{\hat{k}}{2Z_0\sqrt{\mu\epsilon}}\left\{(E_0'^2+E_0''^2)+(E_0'^2-E_0''^2)\cos\left[2\omega\left(t-\frac{n\hat{k}\cdot\vec{r}}{c}\right)\right]+2\vec{E}_0'\cdot\vec{E}_0''\sin\left[2\omega\left(t-\frac{n\hat{k}\cdot\vec{r}}{c}\right)\right]\right\}.$$

The energy propagates along the direction of the unit-vector $\hat{k}=\vec{k}/k$.

Problem 4) a) $\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial\vec{A}}{\partial t} = -\mathrm{Re}\left\{i\psi_0\vec{k}e^{i(\vec{k}\cdot\vec{r}-\omega t)} - i\omega\vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$

$$\Rightarrow \vec{E}(\vec{r},t) = \mathrm{Im}\left\{(\psi_0\vec{k}-\omega\vec{A}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}.$$

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b) $\vec{B}(\vec{r},t) = \mu_0\mu(\omega)\vec{H}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) = \mathrm{Re}\left\{i\vec{k}\times\vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} \Rightarrow$

$$\vec{H}(\vec{r},t) = -\frac{1}{\mu_0\mu(\omega)}\mathrm{Im}\left\{\vec{k}\times\vec{A}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}.$$

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c) 1) $\vec{\nabla}\cdot\vec{D}(\vec{r},t) = \rho_{free}(\vec{r},t) = 0 \Rightarrow \epsilon_0\epsilon\vec{\nabla}\cdot\vec{E}(\vec{r},t) = 0 \Rightarrow \mathrm{Im}\left\{i\vec{k}\cdot(\psi_0\vec{k}-\omega\vec{A}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}=0$

$$\Rightarrow \vec{k}\cdot(\psi_0\vec{k}-\omega\vec{A}_0)=0 \Rightarrow k^2\psi_0 = \omega\vec{k}\cdot\vec{A}_0$$

2) $\vec{\nabla}\times\vec{H}(\vec{r},t) = \vec{J}_{free} + \frac{\partial\vec{D}(\vec{r},t)}{\partial t} \Rightarrow -\frac{1}{\mu_0\mu(\omega)}\mathrm{Im}\left\{i\vec{k}\times(\vec{k}\times\vec{A}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} =$

$\epsilon_0\epsilon(\omega)\mathrm{Im}\left\{-i\omega(\psi_0\vec{k}-\omega\vec{A}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} \Rightarrow$

$\mathrm{Re}\left\{[\vec{k}\times(\vec{k}\times\vec{A}_0) - \mu_0\epsilon_0\mu(\omega)\epsilon(\omega)\omega(\psi_0\vec{k}-\omega\vec{A}_0)]e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0 \Rightarrow$

$(\vec{k}\cdot\vec{A}_0)\vec{k} - k^2\vec{A}_0 - (\omega/c^2)\mu(\omega)\epsilon(\omega)\psi_0\vec{k} + (\omega^2/c^2)\mu(\omega)\epsilon(\omega)\vec{A}_0 = 0 \Rightarrow$

using the result obtained above

$(k^2\psi_0/\omega)\vec{k} - (\omega/c^2)\mu(\omega)\epsilon(\omega)\psi_0\vec{k} = [k^2-(\omega/c)^2\mu(\omega)\epsilon(\omega)]\vec{A}_0 \Rightarrow$

$[k^2-(\omega/c)^2\mu(\omega)\epsilon(\omega)](\psi_0/\omega)\vec{k} = [k^2-(\omega/c)^2\mu(\omega)\epsilon(\omega)]\vec{A}_0$

The last equation is valid for the component of $\vec{A}_0$ parallel to $\vec{k}$. To see this, dot-multiply both sides with $\vec{k}$ and compare with the result obtained from Maxwell's 1st equation earlier. However, for any component of $\vec{A}_0$ that is not parallel to $\vec{k}$, the only way the equation can be satisfied is if $k^2 = (\omega/c)^2 \mu(\omega)\,\epsilon(\omega)$.

3) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \Rightarrow \mathrm{Im}\left\{i\vec{k}\times(\psi_0\vec{k}-\omega\vec{A}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = -\mathrm{Re}\left\{(-i\omega)\,i(\vec{k}\times\vec{A}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$

$\Rightarrow \mathrm{Re}\left\{\left[\vec{k}\times(\psi_0\vec{k}-\omega\vec{A}_0)+\omega(\vec{k}\times\vec{A}_0)\right]e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0 \Rightarrow$

$\vec{k}\times(\psi_0\vec{k}-\omega\vec{A}_0)+\omega\vec{k}\times\vec{A}_0 = 0 \Rightarrow \psi_0\,\cancelto{0}{(\vec{k}\times\vec{k})} - \omega\vec{k}\times\vec{A}_0 + \omega\vec{k}\times\vec{A}_0 = 0$ ✓

Maxwell's 3rd equation is thus automatically satisfied by the choice of $\psi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$.

4) $\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0 \Rightarrow \mathrm{Re}\left\{(i\vec{k})\cdot(i\vec{k}\times\vec{A}_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0 \Rightarrow \vec{k}\cdot(\vec{k}\times\vec{A}_0) = 0$

$\Rightarrow (\vec{k}\times\vec{k})\cdot\vec{A}_0 = 0$ ← Automatically satisfied.

Thus, to satisfy all four equations of Maxwell, we must have $k^2 = (\omega/c)^2\mu(\omega)\,\epsilon(\omega)$ and $k^2\psi_0 = \omega\,\vec{k}\cdot\vec{A}_0$. The latter requirement may also be written as $\vec{k}\cdot\vec{A}_0 = (\frac{\omega}{c^2})\,\mu(\omega)\,\epsilon(\omega)\,\psi_0$.

d) Lorenz gauge: $\vec{\nabla}\cdot\vec{A}(\vec{r},t) + \dfrac{1}{c^2}\dfrac{\partial}{\partial t}\psi(\vec{r},t) = 0 \Rightarrow \mathrm{Re}\left\{i\vec{k}\cdot\vec{A}_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} +$

$\dfrac{1}{c^2}\mathrm{Re}\left\{-i\omega\,\psi_0\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0 \Rightarrow \mathrm{Re}\left\{i(\vec{k}\cdot\vec{A}_0 - \frac{\omega}{c^2}\psi_0)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\} = 0$

$\Rightarrow \vec{k}\cdot\vec{A}_0 - (\frac{\omega}{c^2})\psi_0 = 0 \Rightarrow \vec{k}\cdot\vec{A}_0 = (\frac{\omega}{c^2})\psi_0$ ← Required for the Lorenz gauge.

It is thus seen that the Lorenz gauge is not generally satisfied. Exceptions occur when $\mu(\omega)\,\epsilon(\omega) = 1$, or when $\psi_0 = 0$.