1) $\vec{\nabla}\cdot\vec{S}(r) = \vec{\nabla}\cdot\left[\vec{E}(\vec{r})\times\vec{H}(\vec{r})\right] = \vec{H}(\vec{r})\cdot\vec{\nabla}\times\vec{E}(\vec{r}) - \vec{E}(\vec{r})\cdot\vec{\nabla}\times\vec{H}(\vec{r})$.

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = 0$ in static situations. Also, $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} = \vec{J}$ in magneto-statics. Therefore, $\vec{\nabla}\cdot\vec{S}(\vec{r}) = -\vec{E}(\vec{r})\cdot\vec{J}(\vec{r}) = 0$ ✓

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2) a) $\vec{H}(\vec{r}) = \begin{cases} J_{so}\hat{z}; & 0\le\rho<R_1 \\ 0; & \rho>R_1 \end{cases}$ ← See HW#3, Prob. 6, or HW#4, Prob. 5.

b) Use Gauss' law on a cylindrical surface of radius ρ to find:

$$\vec{E}(\vec{r}) = \begin{cases} 0; & \rho<R_2 \text{ and } \rho>R_1 \\ \frac{R_2\sigma_{so}}{\epsilon_o\rho}\hat{\rho}; & R_2<\rho<R_1 \end{cases}$$

c) $\vec{S}(\vec{r}) = \vec{E}(\vec{r})\times\vec{H}(\vec{r}) = \begin{cases} 0; & \rho<R_2 \text{ and } \rho>R_1 \\ -\frac{R_2\sigma_{so}J_{so}}{\epsilon_o\rho}\hat{\phi}; & R_2<\rho<R_1 \end{cases}$

In the free-space region between the two cylinders, the electromagnetic energy appears to be circulating at a constant rate in the $-\hat{\phi}$ direction.

d) $\vec{\nabla}\cdot\vec{S}(\vec{r}) = \frac{1}{\rho}\frac{\partial S_\phi}{\partial\phi} = 0$ ← Note that this is consistent with Prob. 1 above. The $\vec{E}$-field of the inner cylinder is $\perp$ to $\vec{J}_s$ of the outer cylinder.

Gauss' Theorem: $\oint_{\text{Surface}}\vec{S}\cdot d\vec{s} = \int_{\text{Volume}}(\vec{\nabla}\cdot\vec{S})\,dv = 0$ ✓ ← The closed surface may be inside one or both cylinders, or it may cross their boundaries.

Digression: We will see later in this course that the momentum density of the field is $\vec{S}/c^2$, where $\vec{S}$ is the Poynting vector. The angular momentum $\vec{L}$ of the fields (per unit length of the cylinders) is, therefore,

$$\vec{L} = \int_{\text{volume}}(\rho\hat{\rho})\times(\vec{S}/c^2)\,dv = -\frac{R_2\sigma_{so}J_{so}}{\epsilon_o c^2}\underbrace{\pi(R_1^2-R_2^2)}_{\text{volume}}\hat{z} = -\mu_o\pi R_2(R_1^2-R_2^2)\sigma_{so}J_{so}\hat{z}$$

This angular momentum was created in the beginning, when the current density of the solenoid was raised from 0 to $\vec{J}_{s0}$. During this early period, the magnetic field rose from $\vec{B}=0$ to $\vec{B}=\mu_0 J_{s0}\hat{z}$. According to Maxwell's third equation, $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$; therefore, the $\vec{E}$-field must have been $\vec{E}(\vec{r},t)=-\frac{1}{2}\mu_0\rho\frac{\partial}{\partial t}J_{s0}\hat{\phi}$. This field exerts an azimuthal force on the charge density σ_{s0} on the inner cylinder (radius = R_2), and also on the charge density $-(R_2/R_1)\sigma_{s0}$ that is induced on the inner surface of the solenoid (radius = R_1). The net torque (per unit length) exerted on both cylinders is, therefore, given by:

$$\vec{T}=\int_{z=z_0}^{z_0+1}dz\int_{\phi=0}^{2\pi}\left\{\left(R_2\hat{\rho}\times\sigma_{s0}E_\phi(R_2)\hat{\phi}\right)R_2\,d\phi-\left(R_1\hat{\rho}\times\frac{R_2}{R_1}\sigma_{s0}E_\phi(R_1)\hat{\phi}\right)R_1\,d\phi\right\}$$

$$=-\mu_0\pi R_2^3\sigma_{s0}\frac{\partial J_{s0}}{\partial t}\hat{z}+\mu_0\pi R_2R_1^2\sigma_{s0}\frac{\partial}{\partial t}J_{s0}\hat{z}=\mu_0\pi R_2(R_1^2-R_2^2)\sigma_{s0}\frac{\partial J_{s0}}{\partial t}\hat{z}$$

Since $\vec{T}=d\vec{L}/dt$, the integral of $\vec{T}$ over the period of time that saw an increase in J_{s0} from 0 to its final value, must be equal to the angular momentum imparted to the cylinders by the current source. Thus:

$$\int\vec{T}dt=\mu_0\pi R_2(R_1^2-R_2^2)\sigma_{s0}J_{s0}\hat{z}$$

Conservation of angular momentum then dictates that an equal but opposite angular momentum must reside in the electromagnetic field. ✓

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3) a) $$E_x(\vec{r},t)=E_{x1}+E_{x2}+E_{x3}+E_{x4}=e^{i(k_0\sigma_z z-\omega t)}\Big\{E_{0x}e^{ik_0(\sigma_x x+\sigma_y y)}$$

$$-E_{0x}e^{ik_0(-\sigma_x x+\sigma_y y)}+E_{0x}e^{ik_0(\sigma_x x-\sigma_y y)}-E_{0x}e^{ik_0(-\sigma_x x-\sigma_y y)}\Big\}$$

$$=E_{0x}e^{ik_0(\sigma_z z-ct)}\left[2e^{ik_0\sigma_x x}\cos(k_0\sigma_y y)-2e^{-ik_0\sigma_x x}\cos(k_0\sigma_y y)\right]$$

$$=4iE_{0x}\sin(k_0\sigma_x x)\cos(k_0\sigma_y y)e^{i(k_0\sigma_z z-\omega t)}\ \xrightarrow{\text{Real part}}\ =-4E_{0x}\sin(k_0\sigma_x x)\cos(k_0\sigma_y y)\sin(k_0\sigma_z z-\omega t)$$

$$E_y(\vec{r},t) = E_{oy} e^{ik_o(\sigma_z z - ct)} \left\{ e^{ik_o(\sigma_x x + \sigma_y y)} + e^{ik_o(-\sigma_x x + \sigma_y y)} - e^{ik_o(\sigma_x x - \sigma_y y)} - e^{-ik_o(\sigma_x x + \sigma_y y)} \right\}$$

$$\Rightarrow E_y(\vec{r},t) = -4E_{oy} \cos(k_o\sigma_x x)\sin(k_o\sigma_y y)\sin(k_o\sigma_z z - \omega t)$$

$$E_z(\vec{r},t) = E_{z1} + E_{z2} + E_{z3} + E_{z4} = 4E_{oz} \cos(k_o\sigma_x x)\cos(k_o\sigma_y y)\cos(k_o\sigma_z z - \omega t)$$

At $x = \pm a/2$ the tangential components of the $\vec{E}$-field are E_y, E_z. For a perfect conductor, the tangential $\vec{E}$-field must be zero; therefore,

$$\cos(\pm k_o\sigma_x a/2) = 0 \Rightarrow \cos\left(\frac{\pi a \sigma_x}{\lambda_o}\right) = 0 \Rightarrow \frac{\pi a \sigma_x}{\lambda_o} = m\pi + \frac{\pi}{2} \quad (m = \text{integer})$$

$$\Rightarrow \sigma_x = \left(m + \tfrac{1}{2}\right)\frac{\lambda_o}{a}$$

At $y = \pm b/2$ the tangential components of the $\vec{E}$-field are E_x, E_z. Therefore,

$$\cos\left(\pm k_o\sigma_y \tfrac{b}{2}\right) = 0 \Rightarrow \cos\left(\frac{\pi\sigma_y b}{\lambda_o}\right) = 0 \Rightarrow \sigma_y = \left(n + \tfrac{1}{2}\right)\frac{\lambda_o}{b} \quad n = \text{integer}$$

Any combination of m and n is acceptable so long as $\sigma_x^2 + \sigma_y^2 < 1$; otherwise σ_z will become imaginary, and the beam will not propagate.

b) Surface charge density $\sigma_s = \epsilon_o E_\perp$.

On the walls located at $x = \pm\frac{a}{2} \Rightarrow \sigma_s(x = \pm\frac{a}{2}) = \mp\epsilon_o E_x = \pm 4\epsilon_o E_{ox} \sin(k_o\sigma_x x) \swarrow$

$$\times \cos(k_o\sigma_y y)\sin(k_o\sigma_z z - \omega t) = 4\epsilon_o E_{ox} \sin(m\pi + \tfrac{\pi}{2})\cos(k_o\sigma_y y)\sin(k_o\sigma_z z - \omega t)$$

On the walls located at $y = \pm b/2 \Rightarrow \sigma_s(y = \pm b/2) = \mp\epsilon_o E_y = \pm 4\epsilon_o E_{oy} \cos(k_o\sigma_x x) \swarrow$

$$\times \sin(k_o\sigma_y y)\sin(k_o\sigma_z z - \omega t) = 4\epsilon_o E_{oy} \sin(n\pi + \tfrac{\pi}{2})\cos(k_o\sigma_x x)\sin(k_o\sigma_z z - \omega t)$$

In order to find the surface currents we need to know the $\vec{H}$-field.

$$H_x(\vec{r},t) = H_{x1} + H_{x2} + H_{x3} + H_{x4} = -4H_{ox}\cos(k_o\sigma_x x)\sin(k_o\sigma_y y)\sin(k_o\sigma_z z - \omega t)$$

$$H_y(\vec{r},t) = H_{y1} + H_{y2} + H_{y3} + H_{y4} = -4H_{0y}\sin(k_0\sigma_x x)\cos(k_0\sigma_y y)\sin(k_0\sigma_z z - \omega t)$$

$$H_z(\vec{r},t) = H_{z1} + H_{z2} + H_{z3} + H_{z4} = -4H_{0z}\sin(k_0\sigma_x x)\sin(k_0\sigma_y y)\cos(k_0\sigma_z z - \omega t)$$

Note that on the walls at $x = \pm a/2$, the perpendicular $\vec{H}$-field, H_x, is zero. Similarly, on the walls at $y = \pm b/2$, the $\perp$ field, H_y, is zero, consistent with the absence of $\vec{H}$-field from the interior regions of the metallic conductor, and with the Maxwell equation $\vec{\nabla}\cdot\vec{B} = 0$.

On the walls at $x = \pm a/2 \Rightarrow \vec{J}_s(x = \pm a/2) = \mp 4H_{0z}\sin(k_0\sigma_x x)\sin(k_0\sigma_y y)\cos(k_0\sigma_z z - \omega t)\hat{y}$

$\pm 4H_{0y}\sin(k_0\sigma_x x)\cos(k_0\sigma_y y)\sin(k_0\sigma_z z - \omega t)\hat{z} \Rightarrow$

$$\vec{J}_s(x = \pm a/2) = 4\sin(m\pi + \tfrac{\pi}{2})\left\{-H_{0z}\sin(k_0\sigma_y y)\cos(k_0\sigma_z z - \omega t)\hat{y} + H_{0y}\cos(k_0\sigma_y y)\sin(k_0\sigma_z z - \omega t)\hat{z}\right\}$$

Similarly, on the walls at $y = \pm b/2 \Rightarrow \vec{J}_s(y = \pm b/2) = \pm 4H_{0z}\sin(k_0\sigma_x x)\sin(k_0\sigma_y y)\cos(k_0\sigma_z z - \omega t)\hat{x}$

$\mp 4H_{0x}\cos(k_0\sigma_x x)\sin(k_0\sigma_y y)\sin(k_0\sigma_z z - \omega t)\hat{z} \Rightarrow$

$$\vec{J}_s(y = \pm b/2) = 4\sin(n\pi + \tfrac{\pi}{2})\left\{H_{0z}\sin(k_0\sigma_x x)\cos(k_0\sigma_z z - \omega t)\hat{x} - H_{0x}\cos(k_0\sigma_x x)\sin(k_0\sigma_z z - \omega t)\hat{z}\right\}$$

Note that at the corners the current remains continuous, flowing smoothly from one wall to the adjacent wall.

$$\vec{\nabla}\cdot\vec{J}_s(x = \pm a/2) = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 4k_0\sin(m\pi + \tfrac{\pi}{2})(\sigma_z H_{0y} - \sigma_y H_{0z})\cos(k_0\sigma_y y)\cos(k_0\sigma_z z - \omega t)$$

$$= 4k_0\sin(m\pi + \tfrac{\pi}{2})(E_{0x}/Z_0)\cos(k_0\sigma_y y)\cos(k_0\sigma_z z - \omega t)$$

Also, $\dfrac{\partial \sigma_s(x = \pm a/2)}{\partial t} = -4\varepsilon_0\omega E_{0x}\sin(m\pi + \tfrac{\pi}{2})\cos(k_0\sigma_y y)\cos(k_0\sigma_z z - \omega t)$

Since $k_0/Z_0 = \dfrac{\omega/c}{\sqrt{\mu_0/\varepsilon_0}} = \dfrac{\sqrt{\mu_0\varepsilon_0}}{\sqrt{\mu_0/\varepsilon_0}}\,\omega = \varepsilon_0\omega$, we conclude that $\vec{\nabla}\cdot\vec{J}_s + \dfrac{\partial\sigma_s}{\partial t} = 0$.

The same argument can be used for the walls at $y = \pm b/2$ to prove the conservation of charge.