1) a) $\vec{A}(\vec{r},t) = \dfrac{\mu_0 I_0 d}{4\pi r} \sin[\omega(t - r/c)]\hat{z}$

b) $\vec{\nabla}\cdot\vec{A} + \dfrac{1}{c^2}\dfrac{\partial\psi}{\partial t} = 0 \Rightarrow \dfrac{\partial\psi}{\partial t} = -c^2\,\vec{\nabla}\cdot\vec{A} = -c^2\dfrac{\partial A_z}{\partial z} \Rightarrow$

$$\frac{\partial\psi}{\partial t} = -\frac{1}{\mu_0\epsilon_0}\frac{\mu_0 I_0 d}{4\pi}\frac{\partial}{\partial z}\left\{\frac{\sin(\omega t - k_0 r)}{r}\right\}, \text{ where } r = (x^2+y^2+z^2)^{1/2}.$$

$$\frac{\partial\psi}{\partial t} = -\frac{I_0 d}{4\pi\epsilon_0}\left\{-z(x^2+y^2+z^2)^{-3/2}\sin(\omega t - k_0 r) - \frac{k_0}{r} z (x^2+y^2+z^2)^{-1/2}\cos(\omega t - k_0 r)\right\}$$

$$= \frac{I_0 d}{4\pi\epsilon_0}\left\{\frac{z}{r^3}\sin(\omega t - k_0 r) + k_0\frac{z}{r^2}\cos(\omega t - k_0 r)\right\}$$

$$= \frac{I_0 d\cos\theta}{4\pi\epsilon_0}\left[\frac{1}{r^2}\sin(\omega t - k_0 r) + \frac{k_0}{r}\cos(\omega t - k_0 r)\right\} \leftarrow \cos\theta \text{ is substituted for } z/r.$$

$$\psi(\vec{r},t) = \int\frac{\partial\psi}{\partial t}dt = \frac{I_0 d\cos\theta}{4\pi\epsilon_0}\left[\frac{-1}{\omega r^2}\cos(\omega t - k_0 r) + \frac{k_0}{r\omega}\sin(\omega t - k_0 r)\right]$$

Since $\omega = ck_0$ and $c = 1/\sqrt{\mu_0\epsilon_0}$, we'll have:

$$\psi(\vec{r},t) = \frac{Z_0 I_0 d}{4\pi}\cos\theta\left[\frac{1}{r}\sin(\omega t - k_0 r) - \frac{(\lambda_0/2\pi)}{r^2}\cos(\omega t - k_0 r)\right],$$

which is the same result as obtained in the class.

---

2) a) Limiting form of Bessel functions for small arguments: $J_0(x) \sim 1$, $Y_0(x) \sim \frac{2}{\pi}\ln x$. Therefore, in the near-field where $k_0\rho \ll 1$, we have:

$$\vec{A}(\vec{r},t) \simeq -\frac{1}{4}\mu_0 I_0\left\{\frac{2}{\pi}\ln(k_0\rho)\sin\omega t + \cos\omega t\right\}\hat{z}$$

b) $\vec{B} = \vec{\nabla}\times\vec{A} = -\left(\frac{\partial}{\partial\rho}A_z\right)\hat{\phi}$ ← in Cylindrical Coordinates

$$\Rightarrow \mu_0\vec{H} = \frac{1}{4}\mu_0 I_0\frac{\partial}{\partial\rho}\left\{\frac{2}{\pi}\ln(k_0\rho)\sin\omega t + \cos\omega t\right\}\hat{\phi} = \frac{\mu_0 I_0}{2\pi}\frac{\partial}{\partial\rho}\left[\ln(k_0\rho)\right]\sin\omega t\,\hat{\phi}$$

$$\Rightarrow \vec{H}(\vec{r},t) = \frac{I_0}{2\pi\rho}\sin\omega t\,\hat{\phi}$$

Ampère's law: $\vec{\nabla}\times\vec{H} = \vec{J} \Rightarrow \oint\vec{H}\cdot d\vec{\ell} = I \Rightarrow 2\pi\rho H_\phi = I(t) \Rightarrow I_0\sin\omega t = I_0\sin\omega t$ ✓

STAEDTLER® No. 937 811E Engineer's Computation Pad

c) Sice there are no charges in this problem, $\psi(\vec{r},t)=0$. Therefore,

$$\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}}{\partial t} = \frac{1}{4}\mu_0 I_0 \left\{ \frac{2\omega}{\pi} \ln(k_0\rho)\cos\omega t - \omega \sin\omega t \right\} \hat{z}$$

Replacing ω with ck_0 and $\mu_0 c$ with Z_0, we find:

$$\vec{E}(\vec{r},t) = \frac{1}{4} k_0 Z_0 I_0 \left\{ \frac{2}{\pi} \ln(k_0\rho)\cos\omega t - \sin\omega t \right\} \hat{z}.$$

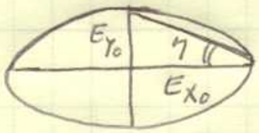
3) a) $\vec{\sigma} = \hat{z}$ ← Homogeneous plane-wave must have real $\vec{\sigma}$, with length 1.

$$\vec{\sigma}\cdot\vec{E}_0 = 0 \Rightarrow \sigma_x E_{x0} + \sigma_y E_{y0} + \sigma_z E_{z0} = 0 \Rightarrow E_{z0} = 0 \Rightarrow \vec{E}_0 = E_{x0}\hat{x} + E_{y0}\hat{y}$$

$$Z_0 \vec{H}_0 = \vec{\sigma}\times\vec{E}_0 = \hat{z}\times(E_{x0}\hat{x} + E_{y0}\hat{y}) = E_{x0}\hat{y} - E_{y0}\hat{x} \Rightarrow \vec{H}_0 = \frac{1}{Z_0}(E_{x0}\hat{y} - E_{y0}\hat{x})$$

b) E_{x0} and E_{y0} must be "in phase" for the beam to be linearly polarized; in other words, if $E_{x0} = |E_{x0}|e^{i\phi_{x0}}$ and $E_{y0} = |E_{y0}|e^{i\phi_{y0}}$, then the condition for linear polarization is $\phi_{x0} = \phi_{y0}$.

c) Let $E_{x0} = |E_{x0}|e^{i\phi_{x0}}$ and $E_{y0} = |E_{y0}|e^{i\phi_{y0}}$. The condition for circular polarization is: $|E_{x0}| = |E_{y0}|$ and $\phi_{x0} - \phi_{y0} = \pm 90°$. In other words, $E_{x0} = \pm i E_{y0}$.

d) 

$\tan\eta = \dfrac{|E_{y0}|}{|E_{x0}|}$ → polarization ellipticity $\eta = \tan^{-1}\dfrac{|E_{y0}|}{|E_{x0}|}$

e) $$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*) = \frac{1}{2}\mathrm{Re}\left\{ \vec{E}_0 \cancel{e^{ik_0(\vec{\sigma}\cdot\vec{r}-ct)}} \times \vec{H}_0^* \cancel{e^{-ik_0(\vec{\sigma}\cdot\vec{r}-ct)}} \right\}$$

$$= \frac{1}{2Z_0}\mathrm{Re}\left\{ (E_{x0}\hat{x} + E_{y0}\hat{y}) \times (-E_{y0}^*\hat{x} + E_{x0}^*\hat{y}) \right\} = \frac{1}{2Z_0}\left(|E_{x0}|^2 + |E_{y0}|^2\right)\hat{z}$$

4) a) $$\vec{E}_1(\vec{r},t) = E_{x1}\hat{x}\exp\{ik_0(y\sin\theta + z\cos\theta - ct)\}$$

$$\vec{E}_2(\vec{r},t) = E_{x2}\hat{x}\exp\{ik_0(-y\sin\theta + z\cos\theta - ct)\}$$

$$\vec{H}_1(\vec{r},t) = \frac{1}{Z_0}\vec{\sigma}_1 \times \vec{E}_1(\vec{r},t) = \frac{E_{x1}}{Z_0}(\cos\theta\,\hat{y} - \sin\theta\,\hat{z})\exp[ik_0(y\sin\theta + z\cos\theta - ct)]$$

$$\vec{H}_2(\vec{r},t) = \frac{1}{Z_0}\vec{\sigma}_2 \times \vec{E}_2(\vec{r},t) = \frac{E_{x2}}{Z_0}(\cos\theta\,\hat{y} + \sin\theta\,\hat{z})\exp[ik_0(-y\sin\theta + z\cos\theta - ct)]$$

b) $$\langle\vec{S}(\vec{r},t)\rangle = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*) = \frac{1}{2}\mathrm{Re}\{(\vec{E}_1+\vec{E}_2)\times(H_1^*+H_2^*)\} =$$

$$\frac{1}{2}\mathrm{Re}(\vec{E}_1\times\vec{H}_1^*) + \frac{1}{2}\mathrm{Re}(\vec{E}_2\times\vec{H}_2^*) + \frac{1}{2}\mathrm{Re}(\vec{E}_1\times\vec{H}_2^*) + \frac{1}{2}\mathrm{Re}(\vec{E}_2\times\vec{H}_1^*) =$$

$$\frac{|E_{x1}|^2}{2Z_0}\hat{x}\times(\cos\theta\,\hat{y} - \sin\theta\,\hat{z}) + \frac{|E_{x2}|^2}{2Z_0}\hat{x}\times(\cos\theta\,\hat{y} + \sin\theta\,\hat{z}) +$$

$$\frac{1}{2Z_0}\mathrm{Re}\left\{E_{x1}e^{ik_0(y\sin\theta + z\cos\theta - ct)}E_{x2}^* e^{-ik_0(-y\sin\theta + z\cos\theta - ct)}\right\}\hat{x}\times(\cos\theta\,\hat{y} + \sin\theta\,\hat{z}) +$$

$$\frac{1}{2Z_0}\mathrm{Re}\left\{E_{x2}e^{ik_0(-y\sin\theta + z\cos\theta - ct)}E_{x1}^* e^{-ik_0(y\sin\theta + z\cos\theta - ct)}\right\}\hat{x}\times(\cos\theta\,\hat{y} - \sin\theta\,\hat{z})$$

$$\Rightarrow \langle\vec{S}(\vec{r},t)\rangle = \frac{|E_{x1}|^2}{2Z_0}(\cos\theta\,\hat{z} + \sin\theta\,\hat{y}) + \frac{|E_{x2}|^2}{2Z_0}(\cos\theta\,\hat{z} - \sin\theta\,\hat{y})$$

$$+ \frac{|E_{x1}||E_{x2}|}{2Z_0}\mathrm{Re}\left\{e^{i(2k_0 y\sin\theta + \phi_{x1} - \phi_{x2})}\right\}(\cos\theta\,\hat{z} - \sin\theta\,\hat{y})$$

$$+ \frac{|E_{x1}||E_{x2}|}{2Z_0}\mathrm{Re}\left\{e^{-i(2k_0 y\sin\theta + \phi_{x1} - \phi_{x2})}\right\}(\cos\theta\,\hat{z} + \sin\theta\,\hat{y})$$

$$\Rightarrow \langle\vec{S}(\vec{r},t)\rangle = \frac{|E_{x1}|^2+|E_{x2}|^2}{2Z_0}\cos\theta\,\hat{z} + \frac{|E_{x1}|^2-|E_{x2}|^2}{2Z_0}\sin\theta\,\hat{y} + \frac{|E_{x1}||E_{x2}|}{Z_0}\cos\theta\cos(2k_0\sin\theta\, y + \phi_{x1} - \phi_{x2})\hat{z}$$

The last term in the above expression is the interference term, modulating the Poynting Vector component $\langle S_z\rangle$ along the y-direction according to the function $\cos(2k_0\sin\theta\, y + \phi_{x1} - \phi_{x2})$. The component $\langle S_z\rangle$ is thus strong when the fringes are bright and weak when the fringes are dark.