Opti 501 **1st Midterm Solutions** (9/28/2021) **Time: 75 minutes**

Problem 1) In the cylindrical system, $\hat{\boldsymbol{\rho}} = (\cos\phi)\hat{\boldsymbol{x}} + (\sin\phi)\hat{\boldsymbol{y}}$. In the spherical system,

$$\hat{\boldsymbol{r}} = (\sin\theta\cos\phi)\hat{\boldsymbol{x}} + (\sin\theta\sin\phi)\hat{\boldsymbol{y}} + (\cos\theta)\hat{\boldsymbol{z}}.$$

$$\hat{\boldsymbol{\theta}} = (\cos\theta\cos\phi)\hat{\boldsymbol{x}} + (\cos\theta\sin\phi)\hat{\boldsymbol{y}} - (\sin\theta)\hat{\boldsymbol{z}}.$$

$$\hat{\boldsymbol{z}} = (\cos\theta)\hat{\boldsymbol{r}} - (\sin\theta)\hat{\boldsymbol{\theta}}.$$

Problem 2) a) Integrating the D-field over the surface of the sphere of radius r yields $4\pi r^2(\varepsilon_0 E)$. This must equal the total charge q inside the sphere. Therefore, $\boldsymbol{E} = q\hat{\boldsymbol{r}}/(4\pi\varepsilon_0 r^2)$.

b) If the E-field vectors happen to be tilted, as depicted in Fig.(b), then standing above the charge makes it appear that the E-field is rotated counterclockwise, whereas standing below the charge would make the direction of rotation appear as clockwise. Since it should not matter whether the observer is on one side or the other, there can be no such tilt.

Note that one cannot make the argument by looking at the charge from one side only. If you say, it appears counterclockwise from above but why not clockwise, then the answer would be: the sign of the charge dictates whether it rotates clockwise or counterclockwise. In other words, there is a difference between positive and negative charges in this regard. However, by looking at the same charge (either positive or negative) from above as well as below, one can see that the symmetry of space is broken and that, therefore, there cannot be a tilt one way or the other.

c) The azimuthal tilt of the E-field as depicted in Fig.(b) would mean that the integral of $\boldsymbol{E}(\boldsymbol{r})$ around a circle of radius r would be nonzero. This violates Maxwell's 3rd equation. Therefore, there cannot be a tilt of $\boldsymbol{E}(\boldsymbol{r})$, clockwise or counterclockwise, away from the radial direction $\boldsymbol{r}$.

Problem 3)
$$\boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}) = \frac{\partial(\rho A_\phi)}{\rho\partial\rho}\hat{\boldsymbol{z}} = \begin{cases} \frac{1}{\rho}\left[\frac{\mathrm{d}(A_0\rho^2)}{\mathrm{d}\rho}\right]\hat{\boldsymbol{z}}; & \rho < R, \\ \frac{1}{\rho}\left[\frac{\mathrm{d}(A_0R^2)}{\mathrm{d}\rho}\right]\hat{\boldsymbol{z}}; & \rho > R. \end{cases} = \begin{cases} 2A_0\hat{\boldsymbol{z}}; & \rho < R, \\ 0; & \rho > R. \end{cases}$$

It is seen that the curl of $\boldsymbol{A}(\boldsymbol{r})$ is zero everywhere outside the cylinder of radius R, whereas it equals the constant field $2A_0\hat{\boldsymbol{z}}$ inside the cylinder.

Problem 4) a) The electric and magnetic fields at the point (r,θ,ϕ) are given by

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} 0; & r < R, \\ Q\hat{\boldsymbol{r}}/(4\pi\varepsilon_0 r^2); & r > R. \end{cases} \quad \text{and} \quad \boldsymbol{H}(\boldsymbol{r}) = \frac{I\hat{\boldsymbol{\phi}}}{2\pi r\sin\theta}.$$

Thus, the total energy density is the sum of the E-field and H-field energy densities, as follows:

$$\mathcal{E}(\boldsymbol{r}) = \tfrac{1}{2}\varepsilon_0 E^2 + \tfrac{1}{2}\mu_0 H^2 = \begin{cases} \dfrac{\mu_0 I^2}{8\pi^2 r^2\sin^2\theta}; & r < R, \\ \dfrac{Q^2}{32\pi^2\varepsilon_0 r^4} + \dfrac{\mu_0 I^2}{8\pi^2 r^2\sin^2\theta}; & r > R. \end{cases}$$

b) Inside the sphere, the Poynting vector is zero (because $\boldsymbol{E} = 0$), whereas in the region outside the sphere we have

$$\boldsymbol{S}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}(\boldsymbol{r}) = -\left(\frac{QI}{8\pi^2\varepsilon_0 r^3 \sin\theta}\right)\hat{\boldsymbol{\theta}} \quad \leftarrow \boxed{S(\boldsymbol{r}) \to \infty \text{ as } \theta \to 0 \text{ and also as } \theta \to \pi.}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{S}(\boldsymbol{r}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, S_\theta) = 0, \qquad \leftarrow (\sin\theta \neq 0;\ \text{that is, } 0 < \theta < \pi).$$

Along the z-axis, the Poynting vector is directed **toward** the wire where $z > R$ and $\theta \to 0$, and **away** from the wire where $z < -R$ and $\theta \to \pi$. Taking a small cylinder of radius ε and height δ, then imagining it placed around the wire at any elevation z along the wire and outside the sphere (i.e., where $|z| > R$), we find the time-rate of flow of electromagnetic energy out of the cylinder (below the sphere) and into the cylinder (above the sphere) to be

$$\pm\underbrace{(2\pi\varepsilon\delta)}_{\text{area of the curved cylinder surface}}\left(\frac{QI}{8\pi^2\varepsilon_0 r^3 \sin\theta}\right) = \pm\frac{QI\delta}{4\pi\varepsilon_0 z^2}. \quad \leftarrow \boxed{r\sin\theta = \varepsilon;\ \text{also } |z| = |r\cos\theta| \cong r}$$

c) In the preceding equation, the product of the current I of the wire, the E-field acting on the wire at z, and the height δ of the small cylinder, corresponds to the integral of $\boldsymbol{E} \cdot \boldsymbol{J}_{\text{free}}$ over a short segment (length $= \delta$) of the wire. Thus, the energy emanates from the wire in the region below the sphere, where the E-field opposes the current, and re-enters the wire in the region above the sphere, where the E-field is aligned with the direction of the current.